Computational Skills Class 3

EC 2000 Modules

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Class 3: Accuracy

- Euler method for solving differential equations.
- Effect of step size on accuracy.
- Underflow error.

Challenge Problem

Simulate the solution to the differential equation:

$$\dot{x} = x^2 \sin(x), \;\; x(0) = 1.33$$

for $0 < t < 5$.

Euler Method

For the differential equation: $\dot{x} = f(x)$

We can write the change in x(t) as

$$x(t+h) = x(t) + \int_{t}^{t+h} f(x(\tau))\ d\tau$$

$$\approx \quad x(t) \quad + \quad hf(x(t))$$

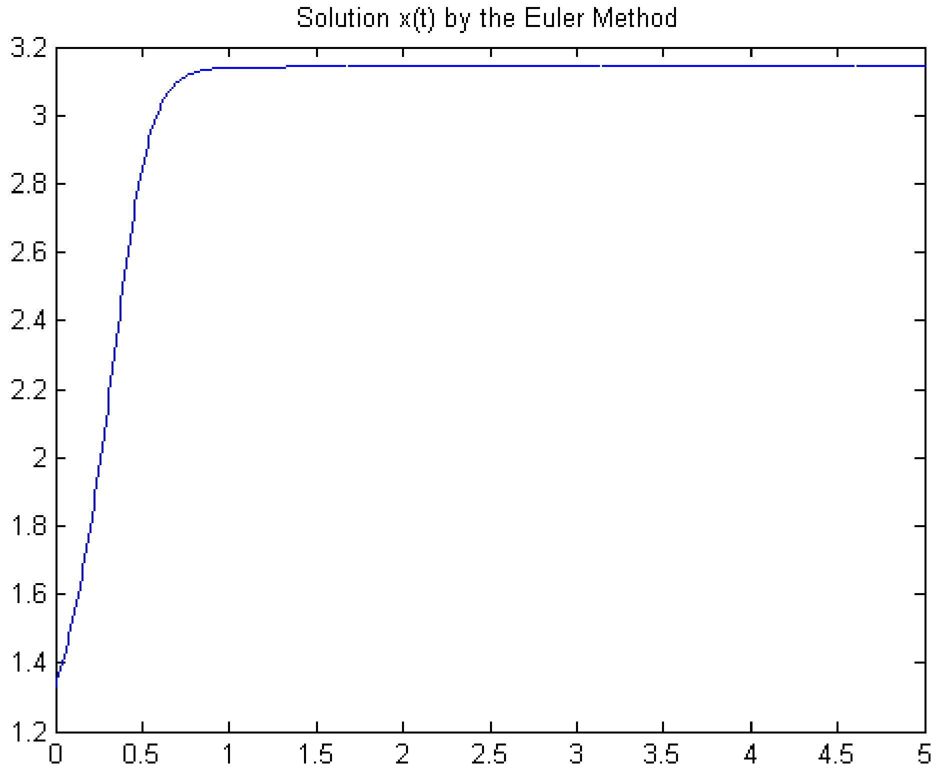
Step size: h

In general, the smaller the step size, the more accurate the simulation will be, but the longer it will take.

A Matlab program to solve the challenge problem is given on the next slide. Note that here we must use looping, and a vector function solution is not possible.

```
% Solve differential equation
x(1) = 1.33;                              % initial condition
h = .01;                                  % step size
for k = 2:501
      x(k) = x(k-1) +h*x(k-1)^2*sin(x(k-1));
end
t = 0:h:5;                                % plot results
plot(t,x)
title('Solution x(t) by the Euler Method')
```

Resulting Graph



Exercise 1

Make a list of all possible sources of inaccuracy in the solution that we just saw.

With the Euler method, we are always making the approximation:

$$\int_t^{t+h} f(x(\tau))\, d\tau \approx x(t) + hf(x(t))$$

So the error added in each step is given by

$$\int_t^{t+h} (f(x(\tau)) - f(x(t))d\tau \approx \frac{df}{dx}(x(t))f(x(t))h^2/2$$

So the fraction of each step that is error is

$$\frac{\int_{t}^{t+h} (f(x(\tau)) - f(x(t))d\tau}{\int_{t}^{t+h} f(x(\tau))d\tau} \approx \frac{h}{2}\frac{df}{dx}(x(t))$$

For a 1% error, we want to keep this to less that .01.

Exercise 2

Select a good step size h to obtain an accuracy of 1% in the solution to the differential equation:

$$\dot{x} = -2\sin(x)$$

Recall: Error fraction is $\frac{h}{2}\frac{df}{dx}(x(t))$

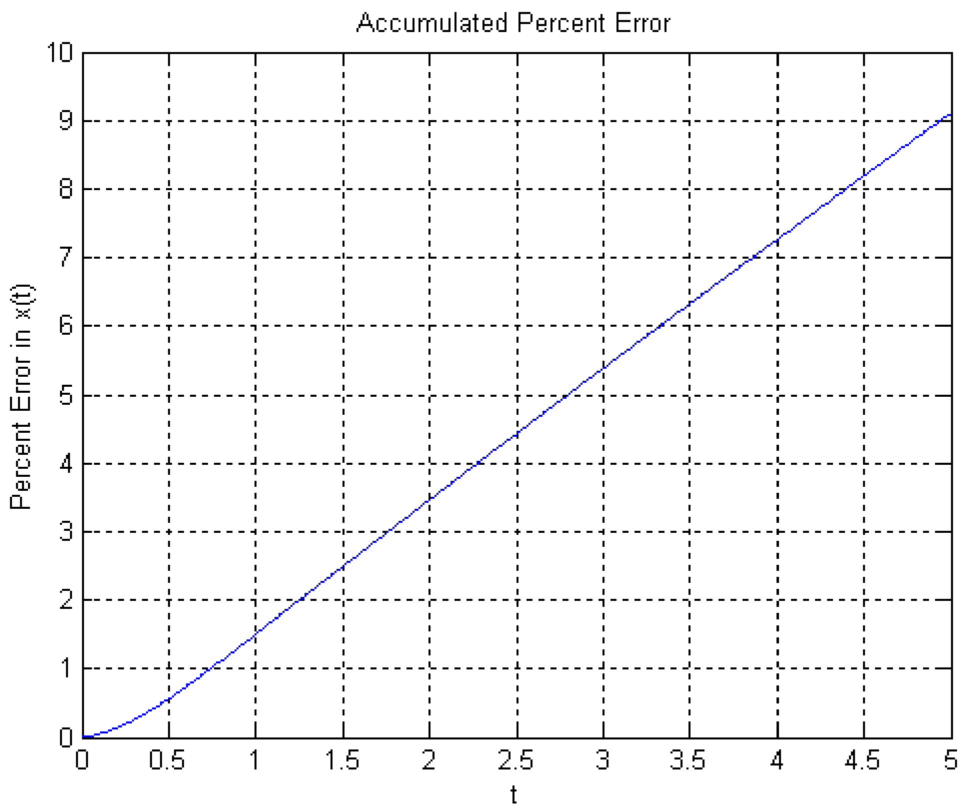
Solution

Fractional error per step (want .01)

$$|error| \leq \frac{h}{2} |-2\cos(x)| \leq h$$

Take $h = .01$

Graph of Error



Note how the percent error increases.

Underflow

Computer arithmetic (floating point) is always approximate. If a very small number ε is added to one on the computer, the result will again be 1, instead of $1+\varepsilon$. This is called an underflow. Underflow errors often occur when adding up a large number of floating point numbers.