# Chapter 11

# STRESS AND DEFORMATION ANALYSIS OF LINEAR ELASTIC BARS IN TENSION

Figure 11.1:

In Chapter10, the equilibrium, kinematic and constitutive equations for a general three-dimensional solid deformable body were summarized. Special cases were also presented for the 2-D and 1-D state of stress. In this chapter we will discuss the application of these 1-D results to the stress and deformation analysis of linear elastic bars in tension.

## 11.1 Necessary Equations for a 1-D Elastic Solid

For a linear elastic solid under *static* equilibrium, we can now summarize the following three sets of equations for any body which has a uniaxial state of stress:

1. **Static Equilibrium Equation** (Conservation of Linear Momentum)

$$\text{x-component: } \frac{d\sigma_{xx}}{dx} + \rho g_x = 0 \tag{11.1}$$

2. **Constitutive Equation for Linear Elastic Isotropic Material**

$$\sigma_{xx} = E\varepsilon_{xx} \tag{11.2}$$

3. **Kinematics** (strain-displacement equation) for small strain

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \tag{11.3}$$

Note that by 1-D we mean that we have a one-dimensional state of stress, i.e., stress is a function of only one position variable (say $x$). However, the body is still three-dimensional but for reasons to be discussed, will generally be restricted in some way with regard to the member geometry (like slender beams, rods, bars, tubes, etc.), cross-sectional area, length, type of loads allowed, etc. Most problems will involve a long, slender geometry. In this and following chapters, we will consider in detail three special cases:

- Bar with axial force only

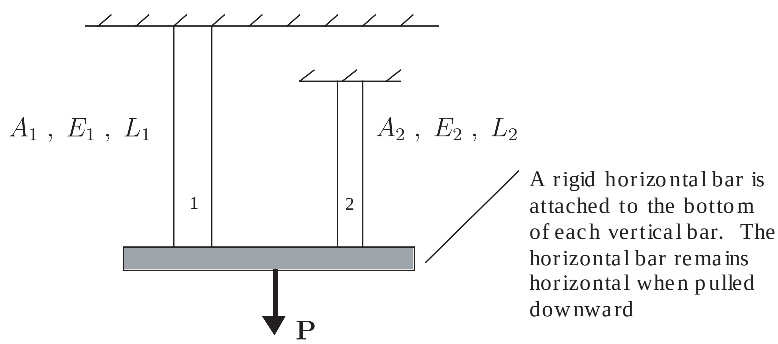



Figure 11.2:

- Bar (or pipe) in torsion
- Beam in bending

In each case, we will develop expressions for the appropriate displacement (or twist) and stress within the body as function of position within the body.

## 11.2 General Solution Procedure

The governing equations for an elastic solid body with a 1-D stress state (function of $x$) are given by equations (11.1)-(11.3). The three equations for a set of coupled ordinary differential equations that must, in general, be solved simultaneously for the stress $\sigma_{xx}$, strain $\varepsilon_{xx}$ and displacement $u_x$. As in the solution of any differential equation, *boundary conditions must be specified to solve this boundary*

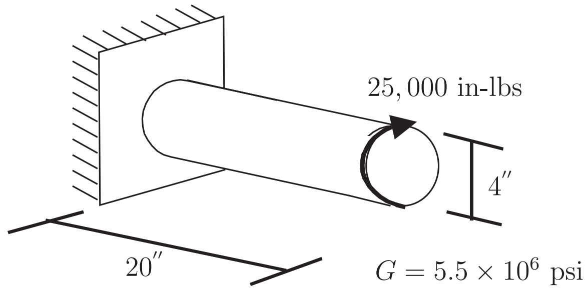



Figure 11.3:

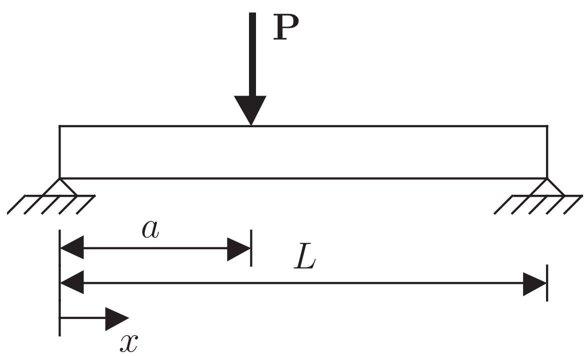



Figure 11.4:

*value problem*. These boundary condition must be *either* displacements or stresses (tractions) on *every* point of the boundary. Consequently, for every problem, we must satisfy the following four (4) sets of equations:

Four Sets of Equations to be Satisfied for A Solid Deformable Body with 1-D Stress State

| | |
|---|---|
| *Static Equilibrium Equation (from COLM)*: | $\frac{d\sigma_{xx}}{dx} + \rho g_x = 0$ |
| *Constitutive Equation (Stress-Strain)*: | $\sigma_{xx} = E\varepsilon_{xx}$ |
| *Kinematics (strain-displacement)*: | $\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$ |
| *Boundary Conditions*: | Depends on problem geometry and loading |

## 11.3 Examples of the 1-D Axial Bar Problems

In this section, we present the solution of a number of problems that are referred to as "uniaxial bar problems." All problems of this type involve members wherein the stress $\sigma_{xx}$ is a uniform normal stress over the cross-section (tension or compression). No shear stress is allowed.

***Example 11-1***

Consider an axial bar as shown below. The bar is of length $L$ and has prismatic cross-sectional area $A$. The left end of the bar is fixed so that it cannot move in the $x$ direction. A force $F$ is applied to the end cross-section on the right.

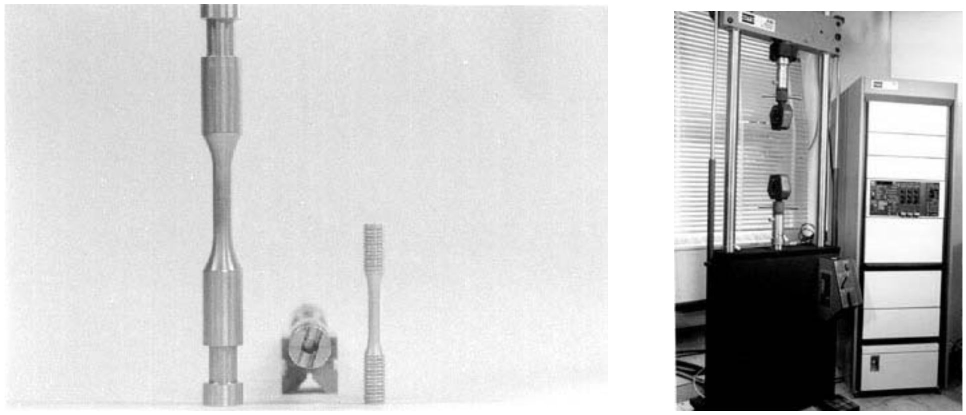


Figure 11.5:

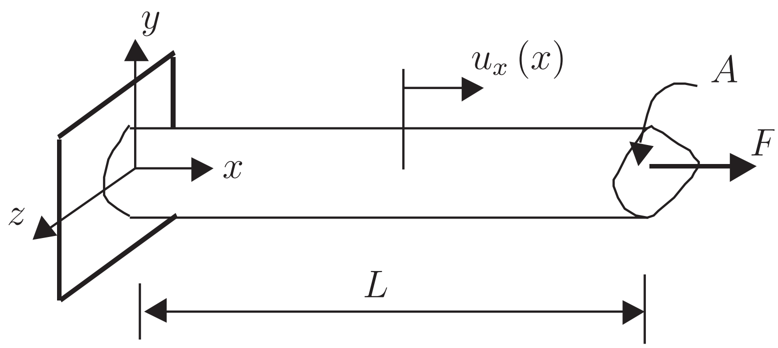



Figure 11.6: Bar Subjected to Axial Force $F$

*Determine*: $u_x(x)$, $\sigma_{xx}(x)$ and $u_x(x = L)$.
*Solution*: We write each of the four required equations:

1) Equilibrium (conservation of linear momentum): $\frac{d\sigma_{xx}}{dx} = 0 \implies \sigma_{xx} = C_1$

2) Constitutive Law: $\sigma_{xx} = E\varepsilon_{xx}$

3) Kinematics (stress-strain): $\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$

4) Boundary Conditions: $u_x(x = 0) = 0$, $\sigma_{xx}(x = L) = \frac{F}{A}$

Integrate the differential equation of equilibrium and evaluate the constant of integration by using the stress boundary condition:

$$\begin{aligned} \frac{d\sigma_{xx}}{dx} = 0 \quad &\implies \quad \sigma_{xx} = C_1 \\ \sigma_{xx}(x = L) = C_1 = \frac{F}{A} \quad &\implies \quad \sigma_{xx}(x) = \frac{F}{A} \end{aligned} \tag{11.4}$$

Combine the kinematics and stress-strain equation to obtain:

$$\frac{\partial u_x}{\partial x} = \varepsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{\frac{F}{A}}{E} \tag{11.5}$$

Write above as:

$$\frac{du_x}{dx} = \frac{F}{AE} \text{ or } \int_0^x du_x = \int_0^x \frac{F}{EA}\,dx$$

and integrate from 0 to $x$ to obtain the axial displacement:

$$u_x(x) = \int_0^x \frac{F}{EA}\,dx = \frac{F}{EA}\int_0^x dx = \left(\frac{F}{EA}\right)x + C \tag{11.6}$$

Apply the displacement boundary condition:

$$u_x(x=0) = \frac{F}{EA}(0) + C \implies C = 0 \tag{11.7}$$

*The solution for the axial bar in tension becomes*:

$$u_x(x) = \left(\frac{F}{EA}\right)x \text{ and } \sigma_{xx}(x) = \frac{F}{A} \tag{11.8}$$

The displacement at end is given

$$\delta_{end} = u_x(x=L) = \frac{FL}{EA} \tag{11.9}$$

Note that *the total elongation of the bar* is also given by this same equation:

$$\delta = \text{elongation} = \frac{FL}{EA} \tag{11.10}$$

This equation is often called the $FLEA$ equation.

***Example 11-2***

Consider the uniaxial bar constructed of two different bars with Young's modulus $E_1$ and $E_2$, respectively, and cross-sectional area $A_1$ and $A_2$, respectively. The bars are fixed between two walls and loaded with a force $P$ applied at point $B$:
*Determine*: The axial force and stress in each bar, and the axial displacement at point $B$.
*Solution*: For this problem, we have 4 relationships to satisfy:

1. *Equilibrium* of horizontal forces at any point
2. *Kinematics* (strain/displacements in horizontal direction)
3. *Stress-Strain* (material properties)
4. *Boundary Conditions*

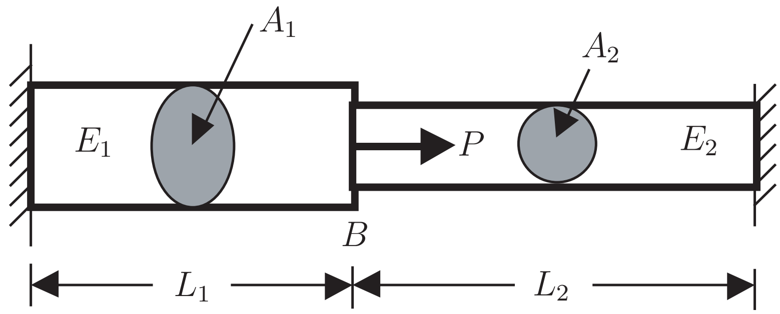



Figure 11.7: Two Bars Fixed Between a Rigid Wall and Loaded with Force $P$

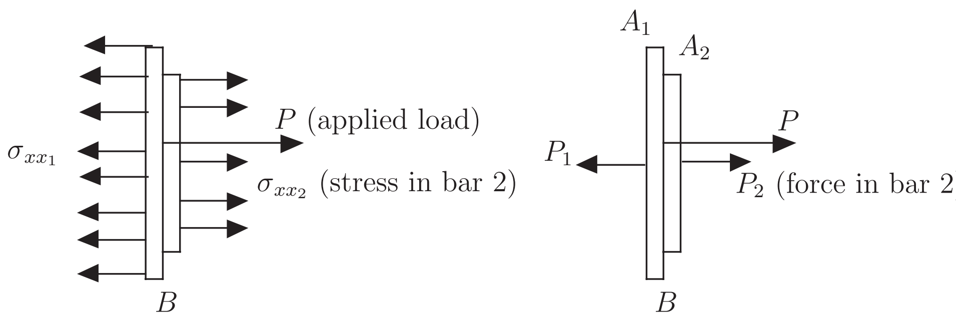



Figure 11.8: Free-Body Diagram of Interface Between Bar 1 and 2

Use a ***"free-body diagram"*** to determine equilibrium of the forces acting on any segment of the bar. Assume that the force in bars 1 and 2 are $P_1$ and $P_2$ (tension), respectively. Take a free-body of the bar cross-section at point $B$ (where the axial load $P$ is applied) to obtain

The stress in each bar can be written in terms of an equivalent force over an area so that the stress in bars 1 and 2 is given by: $P_1 = \int_{A_1} \sigma_{xx_1}\, dA$ and $P_2 = \int_{A_2} \sigma_{xx_2}\, dA =$

*Equilibrium* of the free body at B in terms of forces requires that

$$\sum F_{horizontal} = 0 = P + P_2 - P_1 \implies P = P_1 - P_2 \tag{11.11}$$

Notice that we have only one equilibrium (conservation of linear momentum) equation which can be written, but this equation involves two unknowns: $P_1$ and $P_2$. Hence, equation (11.11) by itself cannot be use to solve for the unknown internal forces $P_1$ and $P_2$. We will need another equation that, in this case, will be a displacement equation based on boundary conditions as shown below.

Any problem for which the internal forces can not be determined by static equilibrium alone is called ***statically indeterminate***. Statically indeterminate structures will always require additional equations (defining displacements) in order to complete the solution.

We know from boundary conditions that the bar's total deformation between the two fixed walls is zero. First, calculate the deformation (elongation) of bars 1 and 2 as if they were taken separately as shown in the free body diagrams shown below:

We know from example 1 that the elongation of a bar is given by equation (11.10)

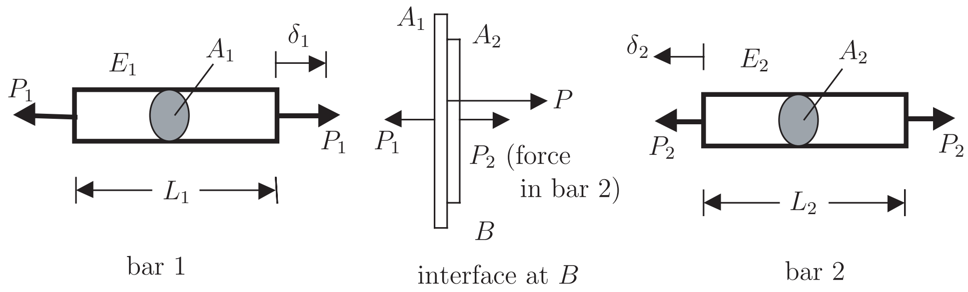



Figure 11.9: Free-Body Diagrams for Bars 1 and 2 and Their Interface

$$\delta_1 = \text{elongation of bar 1} = \frac{P_1 L_1}{E_1 A_1} \tag{11.12}$$
$$\delta_2 = \text{elongation of bar 2} = \frac{P_2 L_2}{E_2 A_2}$$

Note that *equations* (11.12) *implicitly account for equilibrium, constitutive behavior and kinematics* because they were used in deriving equation (11.10).

The *displacement boundary condition* for this problem requires that bars 1 and 2 together have a total elongation of zero (since they are between two fixed walls). Thus, we can write the following:

$$\text{total elongation} = 0 = \delta_1 + \delta_2 = \frac{P_1 L_1}{E_1 A_1} + \frac{P_2 L_2}{E_2 A_2} \tag{11.13}$$

Solve for $P_1$ from equation (11.13) to obtain

$$P_1 = -\frac{E_1 A_1}{L_1}\left(\frac{P_2 L_2}{E_2 A_2}\right) \tag{11.14}$$

From the equilibrium equation (11.11), substitute $P_2 = P_1 - P$ into (11.14) to obtain

$$P_1 = \frac{P}{1 + \left(\frac{A_2 E_2 L_1}{A_1 E_1 L_2}\right)} \tag{11.15}$$

Substitute the last result back into equilibrium (11.11) to obtain $P_2$:

$$P_2 = \frac{-P}{1 + \left(\frac{A_1 E_1 L_2}{A_2 E_2 L_1}\right)} \tag{11.16}$$

The axial stress in each bar is given by:

$$\sigma_{xx_1} = \frac{P_1}{A_1} \text{ and } \sigma_{xx_2} = \frac{P_2}{A_2} \tag{11.17}$$

Note that the stress in the left bar is tensile (since $P_1$ is tensile) while in the right bar the stress is compressive (since $P_2$ is negative).

The displacement of point $B$ (the interface) is given by $\delta_1$ (or $-\delta_2$ since $\delta_2 = -\delta_1$):

$$\delta_B = \delta_1 = \frac{P_1 L_1}{A_1 E_1} = \frac{P(L_1 L_2)}{A_1 E_1 L_2 + A_2 E_2 L_1} \tag{11.18}$$

Consider some examples:

1) Bars are equal length, area and Young's modulus, then $P_1 = -P_2 = \frac{P}{2}$ and we see that each bar carries half of the total applied load.

2) Bars are identical except $A_2 = \frac{A_1}{2}$ (bar 2 is smaller), then $P_1 = \frac{2}{3}P$ (tension) and $P_2 = -\frac{P}{3}$ (compression) and we see that the smaller bar carries less load. The stresses are given by $\sigma_{xx_1} = \frac{P_1}{A_1} = \frac{\frac{2}{3}P}{A_1}$ (tension) and $\sigma_{xx_2} = \frac{P_2}{A_2} = \frac{-\frac{1}{3}P}{\frac{A_1}{2}} = \frac{-\frac{2}{3}P}{A_1}$ (compression) and we see, however, that the stresses are equal and opposite. While bar 2 carries more load, its area is larger; and while bar 1 carries less load, its area is larger; and consequently the stresses are equal in the bars.

3) Bars are identical except $E_2 = \frac{E_1}{2}$ (bar 2 is less-stiff), then $P_1 = \frac{2}{3}P$ and $P_2 = -\frac{P}{3}$ and we see that the less-stiff bar carries less load. The stresses are given by $\sigma_{xx_1} = \frac{P_1}{A_1} = \frac{\frac{2}{3}P}{A_1}$ and $\sigma_{xx_2} = \frac{P_2}{A_2} = \frac{-\frac{1}{3}P}{\frac{A_1}{2}} = \frac{-\frac{2}{3}P}{A_1}$ and we see, however, that the stresses are equal and opposite. While bar 2 carries more load, its area is larger; and while bar 1 carries less load, its area is larger; and consequently the stresses are equal in the bars. Looking at equations (11.14) and (11.16), changing the modulus ratio $\frac{E_2}{E_1}$ has the same effect as changing the area ratio $\frac{A_2}{A_1}$, i.e., we can halve the area of bar 2 or the modulus of bar 2 and the result is same in the amount of force carried by each bar.

***Example 11-3***

Two elastic bars in parallel. Each bar has different properties $A$, $E$ and $L$ as shown. The horizontal bar is rigid and remains horizontal when load is applied.

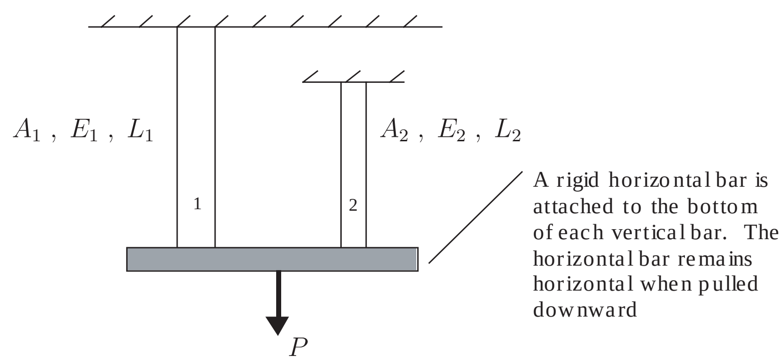



Figure 11.10: Two Parallel Bars

*Determine*: Force, stress and deflection in each bar.

As before, we have the four governing equations that must be satisfied: *Equilibrium*, *Constitutive*, *Kinematics* and *Boundary Conditions*. Assume that the axial force in bars 1 and 2 are $P_1$ and $P_2$ (tension), respectively. Use a ***"free-body diagram"*** to determine equilibrium of the forces acting in each bar. Take a free-body by cutting each bar below its fixed point:

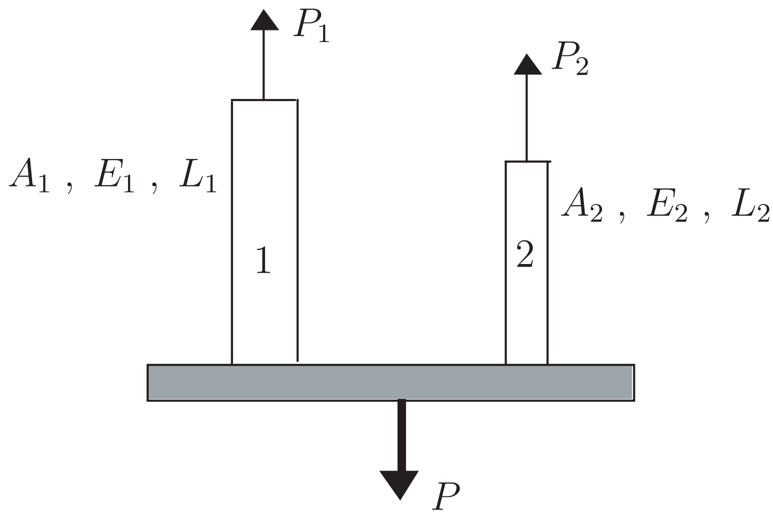



Figure 11.11: Free-Body for Two Parallel Bar System

*Equilibrium* of the free body in terms of forces requires that

$$\sum F_{vertical} = 0 = P_1 + P_2 - P \implies P = P_1 - P_2 \tag{11.19}$$

The elongation of each bar is given

$$\begin{aligned} \delta_1 = \quad & \text{elongation of bar 1} \quad = \frac{P_1 L_1}{E_1 A_1} \\ \delta_2 = \quad & \text{elongation of bar 2} \quad = \frac{P_2 L_2}{E_2 A_2} \end{aligned} \tag{11.20}$$

As stated in the previous example, *equations* (11.20) *implicitly account for equilibrium, constitutive behavior and kinematics* because they were use in deriving equation (11.10).

Assume the horizontal bar moves down a distance $\delta$ when the load $P$ is applied. Since the horizontal bar is required to remain horizontal, then the elongation of each of the vertical bars must be equal to the movement $\delta$ of the horizontal bar. Thus our *displacement boundary condition* can be written:

$$\delta_1 = \delta_2 = \delta = \left( \begin{array}{c} \text{movement of} \\ \text{horizontal bar} \end{array} \right) = \frac{P_1 L_1}{E_1 A_1} = \frac{P_2 L_2}{E_2 A_2} \tag{11.21}$$

We now have two equations [(11.19) and (11.21)] and two unknowns ($P_1$ and $P_2$). Writing the two equations in matrix notation:

$$\left[ \begin{array}{cc} 1 & 1 \\ \frac{L_1}{(A_1 E_1)} & -\frac{L_2}{(A_2 E_2)} \end{array} \right] \left\{ \begin{array}{c} P_1 \\ P_2 \end{array} \right\} = \left\{ \begin{array}{c} P \\ 0 \end{array} \right\} \tag{11.22}$$

*Solving for* $P_1$ *and* $P_2$ from the above gives

$$P_1 = \frac{\begin{vmatrix} P & 1 \\ 0 & -\frac{L_2}{(A_2E_2)} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ \frac{L_1}{(A_1E_1)} & -\frac{L_2}{(A_2E_2)} \end{vmatrix}} = \frac{\frac{PL_2}{(A_2E_2)}}{\frac{L_2}{(A_2E_2)} + \frac{L_1}{(A_1E_1)}} = \frac{P}{1 + \left(\frac{A_2E_2L_1}{A_1E_1L_2}\right)} \tag{11.23}$$

$$P_2 = \frac{\begin{vmatrix} 1 & P \\ \frac{L_1}{(A_1E_1)} & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ \frac{L_1}{(A_1E_1)} & -\frac{L_2}{(A_2E_2)} \end{vmatrix}} = \frac{P}{1 + \left(\frac{A_1E_1L_2}{A_2E_2L_1}\right)} \tag{11.24}$$

*Stress* in each bar is given by

$$\sigma_1 = \frac{P_1}{A_1} = \frac{\frac{P}{A_1}}{1 + \left(\frac{A_2E_2L_1}{A_1E_1L_2}\right)} \tag{11.25}$$

$$\sigma_2 = \frac{P_2}{A_2} = \frac{\frac{P}{A_2}}{1 + \left(\frac{A_1E_1L_2}{A_2E_2L_1}\right)} \tag{11.26}$$

The *deflection* of each bar is equal and given by substituting $P_1$ or $P_2$ into (11.21) to obtain

$$\delta_1 = \left(\frac{P_1L_1}{A_1E_1}\right) = \delta_2 = \frac{PL_1L_2}{A_1E_1L_2 + A_2E_2L_1} \tag{11.27}$$

Consider some examples:

1) Bars are equal length, area and Young's modulus, then $P_1 = P_2 = \frac{P}{2}$ (tension) and we see that each bar carries half of the total applied load.

2) Bars are identical except $A_2 = \frac{A_1}{2}$ (bar 2 is smaller), then $P_1 = \frac{2}{3}P$ (tension) and $P_2 = \frac{P}{3}$ (tension) and we see that the smaller bar carries less load. The stresses are given by $\sigma_{xx_1} = \frac{P_1}{A_1} = \frac{\frac{2}{3}P}{A_1}$ and $\sigma_{xx_2} = \frac{P_2}{A_2} = \frac{\frac{1}{3}P}{\frac{A_1}{2}} = \frac{\frac{2}{3}P}{A_1}$ and we see, however, that the stresses are equal and in tension. While bar 1 carries more load, its area is larger; and while bar 2 carries less load, its area is smaller; and consequently the stresses are equal in the bars.

### *Example 11-4*

Uniaxial elastic bar subjected to a uniform temperature increase of $\Delta T$:
The bar is fixed between two wall and has a constant cross-section $A$.
*Determine*: axial strain and stress in the bar and the force on the wall.
For this problem, we have 4 governing equations to satisfy:
*Equilibrium*: $\frac{\partial \sigma_{xx}}{\partial x} = 0$
*Constitutive (Stress-Strain)*: $\sigma_{xx} = E\varepsilon_{xx}^{elastic}$
*Kinematics*: $\varepsilon_{xx}^{total} = \varepsilon_{xx}^{elastic} + \varepsilon_{xx}^{thermal}$, $\varepsilon_{xx}^{thermal} = \alpha\Delta T$, and $\varepsilon_{xx}^{total} = \frac{\partial u_x}{\partial x}$ = axial strain measured/observed (e.g., by a strain gage)
*Boundary Conditions*: $u_x(x=0) = 0$ & $u_x(x=L) = 0$ or $\varepsilon_{xx}^{total} = \frac{\partial u_x}{\partial x} = 0$

### *Solution*

1) $\varepsilon_{xx}^{total} = \frac{\partial u_x}{\partial x} = 0$

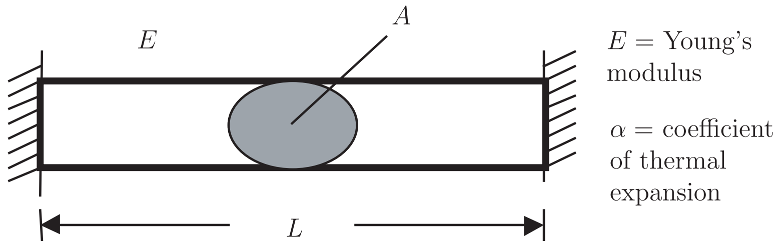



Figure 11.12:

2) Combine constitutive, kinematics, and boundary condition to obtain

$$\sigma_{xx} = E\varepsilon_{xx}^{elastic} = E(\varepsilon_{xx}^{total} - \varepsilon_{xx}^{thermal}) = E(-\alpha\Delta T) = -E\alpha\Delta T$$

3) $\sigma_{xx} = \frac{P}{A} \implies P = \sigma_{xx}A = (-E\alpha\Delta T)A = -E\alpha\Delta TA$ (bar is in compression)

Suppose we have an aluminum bar of area $A = 0.1$ in$^2$ with $E = 10 \times 10^6$ psi and $\alpha = 6 \times 10^{-6}\ \frac{\left(\frac{\text{in}}{\text{in}}\right)}{°\text{F}}$ and thermal loading of $\Delta T = 250$ °F. Then $\sigma_{xx} = -15000$ psi and $P = -1500$ lb (compression).

***Example 11-5***

Assume a ***long*** bar with end load $F$ and whose cross-sectional area is a function of $x$: $A = A(x)$.

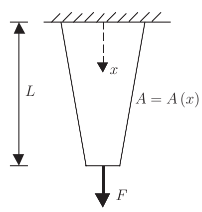



Figure 11.13:

Stress-Strain: $\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$
Kinematics: $\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$
Equilibrium at x: $T_{xx}(x) = \frac{F}{A(x)}$ Displace B.C.: $u_x(x = 0) = 0$
Combine all equations to obtain $\frac{\partial u_x}{\partial x} = \varepsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{\left(\frac{F}{A}\right)}{E}$ or $u_x(x) = \int_0^x \frac{F}{AE}\,dx = \frac{F}{E}\int_0^x \frac{1}{A}\,dx$
The displacement at end is given by: $\delta_{end} = u_x(x = L)$

***Example 11-6***

Axial force at any cross-section not constant, $F = F(x)$.

Same as above except leave $F$ inside of the integral to obtain: $u_x(x) = \int_0^x \frac{F}{AE}\,dx$. $F$ can be a function of $x$ if gravity acts in the $x$ direction.

***Example 11-7***

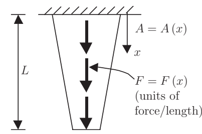



Figure 11.14:

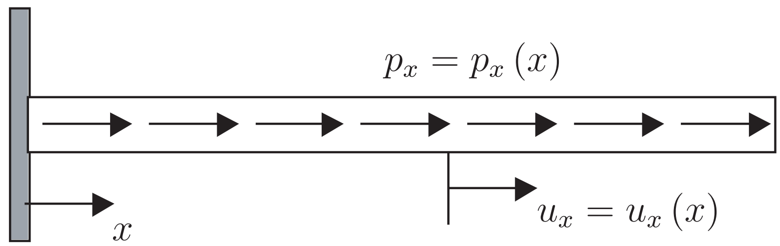



Figure 11.15:

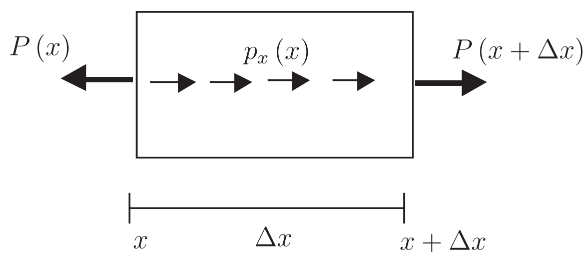



Figure 11.16:

A distributed axial load $p_x(x)$ [units of $\frac{\text{force}}{\text{length}}$] is applied to an axial bar as shown below:

In order to obtain the equilibrium equation, consider a free-body of the bar at point $x$ as shown below. Assume the force in the bar at point $x$ is $P(x)$ and at $x + \Delta x$ is $P(x + \Delta x)$.

For equilibrium in $x$ direction:

$$P(x + dx) - P(x) + p_x\, dx = 0.$$

Divide by $dx$ and take limit to obtain the *equilibrium equation*:

$$\frac{\partial P}{\partial x} + p_x = 0 \tag{11.28}$$

Recall $P = \sigma_{xx} A = (E\varepsilon_{xx})A = EA\frac{\partial u_x}{\partial x}$. Substitute $P$ into the equilibrium equation to obtain the equilibrium equation in terms of axial displacement:

$$\frac{\partial}{\partial x}\left(EA\frac{\partial u_x}{\partial x}\right) + p_x = 0 \tag{11.29}$$

### *Example 11-8*

The concrete pier of square cross section is 8 m high. The sides taper uniformly from a width of 0.5 m at the top to 1.0 m at the bottom. Determine the shortening of this pier due to the compressive load shown. Take the modulus of elasticity of concrete as $2 \times 10^{10}\ \frac{\text{N}}{\text{m}^3}$.

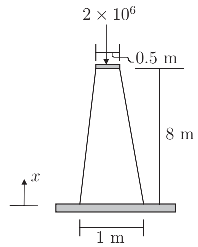



Figure 11.17:

### *Solution*

We use the general solution presented in Example 11.6: $u_x(x) = \int_0^x \frac{P(x)}{EA(x)}\, dx$ which should be reasonable for an angle of taper of 18°.

Let the width of the pier be $w(x)$ at any distance $x$ from the bottom, then $A(x) = w^2(x)$.

Note that $w = 1$ m at $x = 0$ and $w = 0.5$ m at $x = 8$. Since the side is a straight line, then the width varies linearly with $x$. Assume the width is given by $w(x) = ax + b$. To solve for $a$ and $b$, se use the known width at $x = 0$ and $x = 8$ to obtain $b = 1$ and $a = -0.0625$. Hence,

$$\begin{aligned} w(x) &= -0.0625x + 1 \text{ m} \\ A(x) &= (1 - 0.0625x)^2 \text{ m}^2 \end{aligned}$$

Now $P(x)$ is constant throughout the pier (neglecting gravity) so that $P = 2 \times 10^6$ N. Substituting $P$, $E$ and $A(x)$ into the displacement equation gives:

$$\begin{aligned} \therefore u_x(x) &= \int_0^x \frac{2 \times 10^6}{2 \times 10^{10}(1 - 0.0625x)^2}\, dx \quad \text{(units of m)} \\ &= 10^{-4} \int_0^x \frac{1}{(1 - 0.0625x)^2}\, dx = \frac{10^{-4}}{(0.0625)} \left[ \frac{(-1)(-0.0625)}{1 - 0.0625x} \right]_0^L \\ &= \frac{10^{-4}}{(0.0625)} (-0.0625) \left[ 1 - \frac{1}{1 - 0.0625L} \right] \end{aligned}$$

Shortening of pier $= -u_x(x = L) = -\frac{(10^{-4})}{0.0625}(-0.0625)\left[1 - \frac{1}{0.5}\right] = 6.25 \times 10^{-4}$ cm $= 0.16$ cm

### *Example 11-9*

*Given*: A prismatic bar of length 10 ft and cross-section area of 6 in$^2$ , carries a distributed axial load of 5 $\frac{\text{lb}}{\text{ft}}$ and end load of 3000 lb. as shown:

*Required*: Determine Young's Modulus so that the bar will elongate:

a) No more than 2 inches

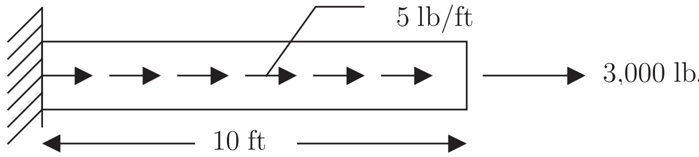



Figure 11.18:

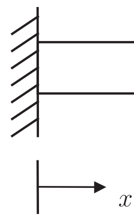



Figure 11.19:

b) No more than 0.5 inch

*Solution*

$$
\begin{aligned}
\frac{d}{dx}\left(AE\frac{du_x}{dx}\right) &= -p_x \\
AE\frac{du_x}{dx} &= -\frac{5}{12}x + c_1 \\
\frac{du_x}{dx} &= -\frac{5}{6(12)E}x + \frac{c_1}{6(E)} \\
u_x(x) &= -\frac{5x^2}{144E} + \frac{c_1 x}{6E} + c_2 \\
AE\frac{du_x}{dx}(x=120) = 3000 &= -\tfrac{5}{12}(120) + c_1 \\
3050 &= c_1 \\
u_x(x=0) = 0 &= c_2 \\
u_x(x) &= -\frac{5}{144}\frac{x^2}{E} + \frac{3050x}{6E}
\end{aligned}
$$

a) $u_x(120) = 2$ in. $\quad 2 = -\frac{5}{144}\frac{(120)^2}{E} + \frac{3050(120)}{6E} \implies$ E $\geq$ 30250 psi

b) $u_x(120) = 0.5$ in. $\quad 0.5 = -\frac{5}{144}\frac{(120)^2}{E} + \frac{3050(120)}{6E} \implies$ E $\geq$ 121000 psi

***Example 11-10***

*Given*: A rod as shown:
*Assume*:

1) Mass density $\rho = 5000\ \frac{\text{kg}}{\text{m}^3}$

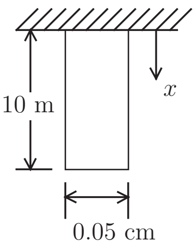



Figure 11.20:

2) $E = 150 \times 10^9$ Pa

*Required*: Determine the elongation of the rod due to its own weight.
*Solution*:

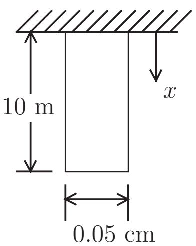



Figure 11.21:

$$W = \rho V_g = \rho A h g \implies W(x) = \rho A g x$$

$$W = 5000(9.8)Ax$$

$$W = 49000Ax$$

$$\frac{d}{dx}\left(AE\frac{du_x}{dx}\right) = -g_x = \rho A g$$

$$AE\frac{du_x}{dx} = \rho A g x + c_1$$

*B.C.*

$$AE\frac{du_x}{dx}(x = 10) = W(x = 10) = \rho A g(10) + c_1 = \rho A g(10)$$

$$\therefore c_1 = 0$$

$$\begin{aligned}
u_x(x) &= \frac{\rho A g x^2}{2}\frac{1}{AE} + c_2 \\
u_x(x) &= \frac{\rho g x^2}{2E} + c_2 \\
u_x(x=0) &= 0 c_2 = 0 \\
u_x(x) &= \frac{\rho g x^2}{2E} = \frac{5000(9.8)x^2}{2(150 \times 10^9)}
\end{aligned}$$

$$u_x(x=10) = 1.63 \times 10^{-5} \text{ m}$$

# Deep Thought

Some people go a little too far in their study of elasticity.

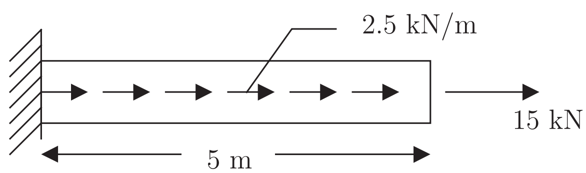
## 11.4 Questions

11.1 Consider the stress tensor for uniaxial load. If a load is applied in the $x$ direction, then find the components of the stress tensor? Use a symbolic variable for those components of stress, which are not zero and use a $3 \times 3$ matrix array.

11.2 Compute the strain tensor for the elastic bar under uniaxial load in 11.1 indicating the components in terms of strain variable. Specify the differences between the stress and strain tensor.

## 11.5 Problems

11.3 *GIVEN*: A prismatic bar is loaded as shown:




Problem 11.3

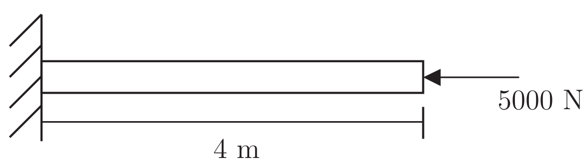
*ASSUME*: Cross sectional area, $A = 6$ in$^2$

*REQUIRED*: Determine the Young's Modulus so that the bar will elongate:

a) No more than 2 inches

b) No more than .5 inch

11.4 *GIVEN*: A prismatic bar is loaded as shown below.




Problem 11.4

*ASSUME*: Cross-sectional area of 25 cm$^2$.

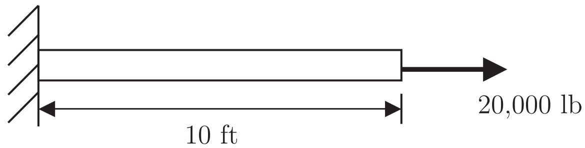
*REQUIRED*: Determine the necessary Young's modulus such that the bar will deflect no more than 0.01 m.

11.5 *GIVEN*: A prismatic bar is loaded as shown below.

*ASSUME*: The bar is made of steel with $E = 29 \times 10^6$ psi.

*REQUIRED*:

a) Determine the necessary cross-sectional area such that the bar will deflect no more than 0.01 in.



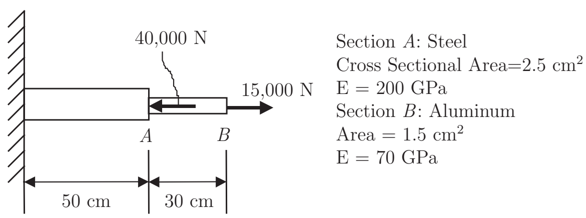

Problem 11.5

b) Determine the axial stress and strain in the bar. If the steel "yields" (becomes inelastic) at 30,000 psi of stress, should we be concerned about possible failure and why?

11.6 For the given data, compute the following:

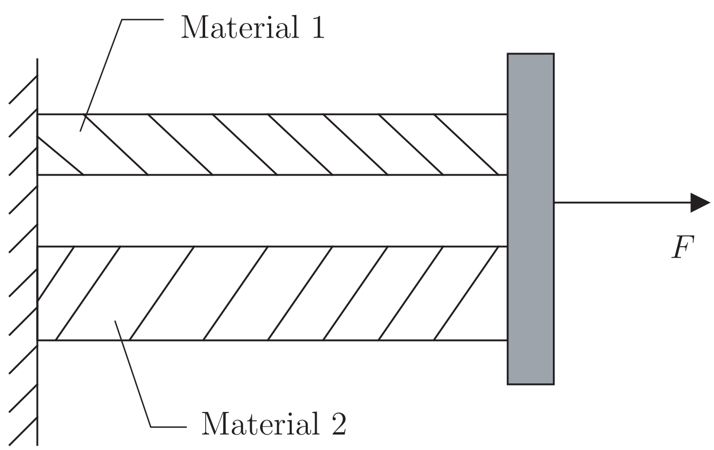
a) Horizontal displacement of point B on the bar.

b) Stress and strain in each bar.




Problem 11.6

11.7 *GIVEN*: The configuration below shows two parallel bars with different properties. The vertical bar to which the force is applied is rigid and constrained to remain vertical.

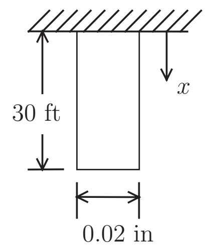



Problem 11.7

$$
\begin{aligned}
L_1 = L_2 &= 10 \text{ m} \\
E_1 &= 10 \text{ GPa} \\
E_2 &= 50 \text{ GPa} \\
A_1 &= 2 \text{ cm}^2 \\
A_2 &= 4 \text{ cm}^2 \\
\sigma_y^1 &= 35 \text{ MPa} \\
\sigma_y^2 &= 100 \text{ MPa}
\end{aligned}
$$

*FIND*: The maximum force, $F$, so that neither material exceeds its yield stress.

11.8 *GIVEN*: A circular prismatic rod as shown:




Problem 11.8

*ASSUME*:

1) Mass density $\rho = 300 \ \frac{\text{lb}}{\text{ft}^3}$
2) $E = 22 \times 10^6$ psi

*REQUIRED*:

Determine the

a) Elongation of the rod due to its own weight.
b) Axial stress as a function of position $x$.

11.9 *GIVEN*: A uniaxial bar with two different cross-sections, $A_1$ and $A_2$, as shown below. The material constants are the same for both parts of the bar.

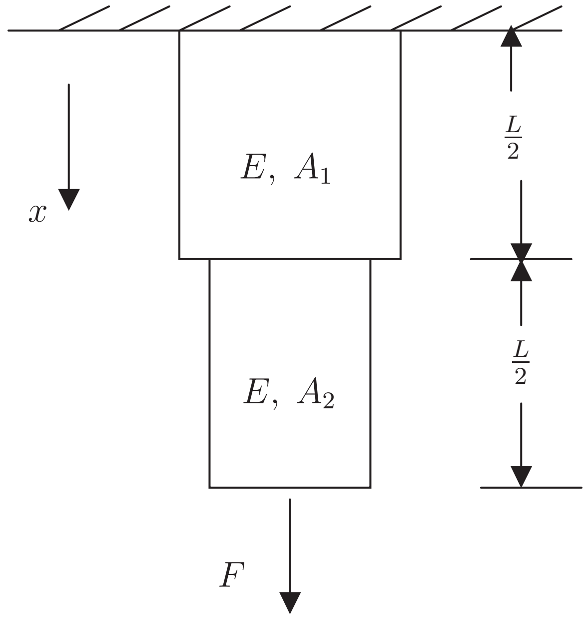
*REQUIRED*:

(a) Calculate the stress and strain in each cross-section; and
(b) Find the deflection at the free end of the bar.

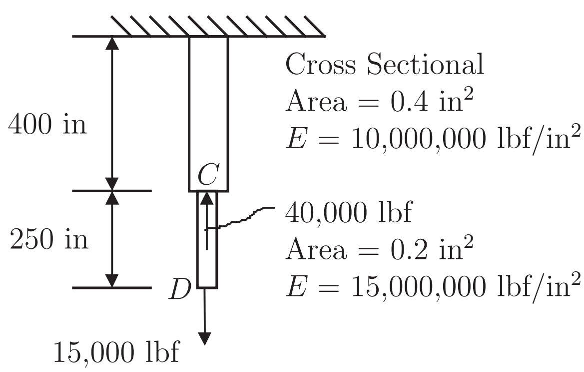
11.10 Compute the following:

a) Vertical movement of point D on the lower bar.
b) Stress in each bar.

11.11 A composite bar consists of steel joined to aluminum as shown. The bar is securely supported at both ends and resists the axial forces shown. Determine the following:




Problem 11.9




Problem 11.10

a) Axial (horizontal) component of reaction force at each end of the bar

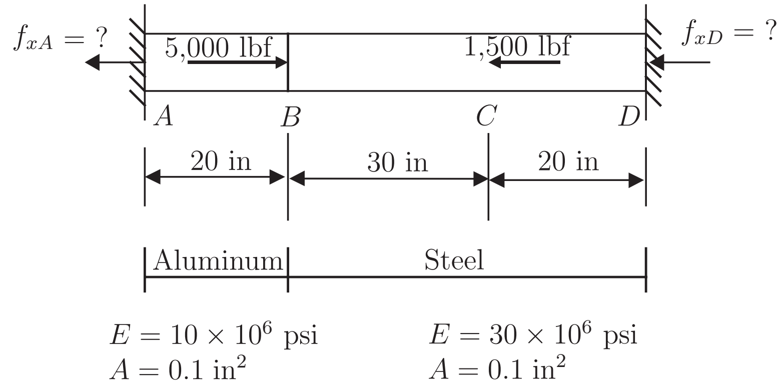
b) Axial displacement of point B

c) Stress and strain in each bar. Note: the stress in bar BD may not be constant.

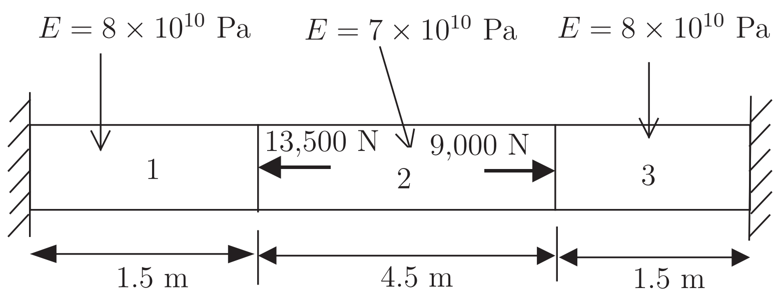
11.12 *GIVEN*: A bar with circular cross-section is fixed at both ends and different Young's modulus as shown.

*ASSUME*: Cross-sectional area, $A = 13\ \text{cm}^2$ (uniform).

*REQUIRED*: Determine the following:




Problem 11.11




Problem 11.12

a) The displacements at the fixed ends
b) The displacements at the *center* of the bar
c) The reactions at the fixed ends
d) The stress and strain in each bar.

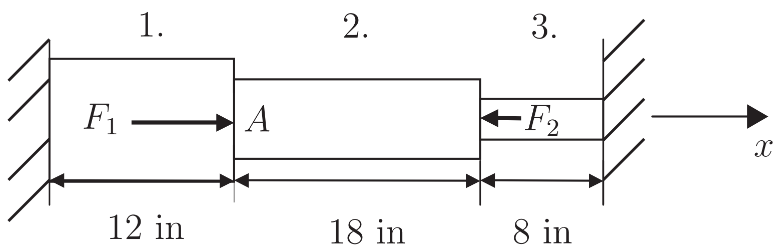
11.13 *GIVEN*: A varying cross-section rod as shown below.

*ASSUME*:

1) The ends are fixed.
2) $E_1 = 5 \times 10^6$ psi ; $E_2 = 15 \times 10^6$ psi ; $E_3 = 10 \times 10^6$ psi
3) $A_1 = 10$ in$^2$ ; $A_2 = 6$ in$^2$ ; $A_3 = 3$ in$^2$
4) $F_1 = 2000$ lb$_\mathrm{f}$ ; $F_2 = 1000$ lb$_\mathrm{f}$

*REQUIRED*:

a) Determine the reaction forces at each end.



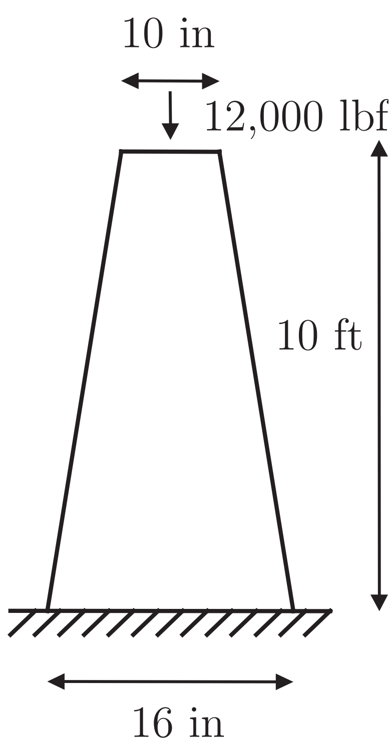

Problem 11.13

b) Determine the displacements at each end.

c) Displacement at point A

d) Stress and strain in each bar.

11.14 *GIVEN*: A column support for a bridge as shown:




Problem 11.14

*ASSUME*:

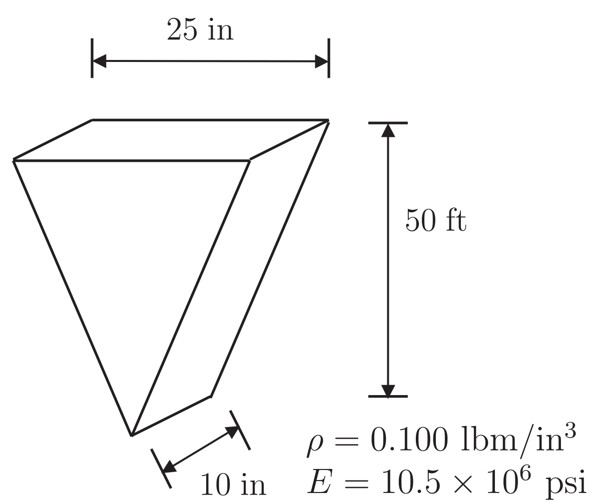
1) Square cross-section
2) $E^L = 12 \times 10^6$ psi

*REQUIRED*: Determine the displacement at:

a) The bottom of the support
b) The middle of the support
c) The top of the support

Hint: $\int (a + bx)^n \, dx = \frac{(a+bx)^{n+1}}{(n+1)b}$ for $n \neq -1$

11.15 *GIVEN*: A wedge-shaped aluminum part as shown below which is fixed from movement at its upper surface.




Problem 11.15

*ASSUME*:

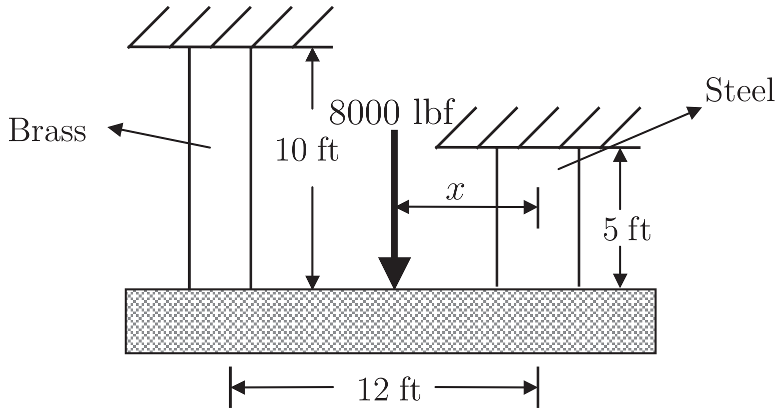
1) $\rho = 0.100 \, \frac{\text{lb}_\text{m}}{\text{in}^3}$
2) $E = 10.5 \times 10^6$ psi

*REQUIRED*: Determine the elongation of the part due to its own weight.

Hint: Consider gravity and determine weight as a function of vertical position.

11.16 *GIVEN*: Two vertical prismatic rods as shown below horizontally suspend a stiff bar of negligible weight. One of the rods is made of steel with a Young's modulus of $30 \times 10^6$ psi and a length of 5 ft while the other is made of brass with a Young's modulus $14 \times 10^6$ psi and a length of 10 ft.

*REQUIRED*: In order for the bar to remain horizontal, where must a load of 8000 $\text{lb}_\text{f}$ be placed (i.e., position $x$)? Assume the bars behave elastically and that the bending of the horizontal bar is negligible.




Problem 11.16

11.17 A load $P = 1000$ N is applied to bar #1 at point $A$ according to the diagram shown below. The horizontal bar is rigid and remains horizontal when the load is applied. Considering that only vertical displacement is allowed determine the following:

a) The reaction forces at points D and F.
b) The displacements of points C and A.
c) The strain in each bar (1, 3 & 4).
d) The stress in each bar (1, 3 & 4).

11.18 In order to complete the design for the press shown below; determine the required cross sectional area of element #3. The properties for the elements are given as follows: Elements #1 and #2 are made of steel (elastic modulus of 200 GPa) while element #3 is made of aluminum (elastic modulus of 70 GPa). The cross sectional area for element #1 is 2.0 cm$^2$ while for #2 is 2.5 cm$^2$. Finally, the length of element #1 is 15 cm and 25 cm for elements #2 and #3. The 100 kN force is applied to the end of bar 1.

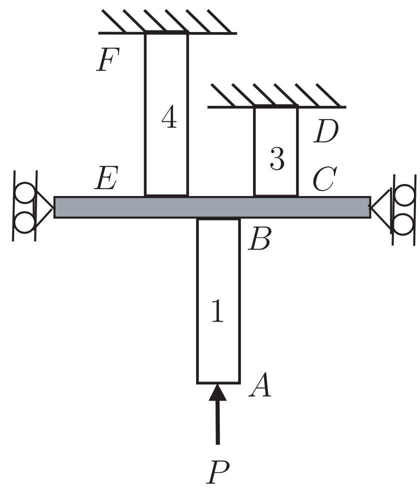
Note: Observe that the plate connecting all elements is constrained so that no rotation is allowed (i.e. it maintains a horizontal position at all times). Also note that the load $P$ is applied directly to element #1.

11.19 For the design problem stated above in 11.18, if all elements have a square cross section, then calculate the selected width of element #3. Also calculate the corresponding strain and stress acting on each element.

11.20 *GIVEN*: A bar with cross-sectional area $A = 5$ in$^2$ of is loaded as shown:

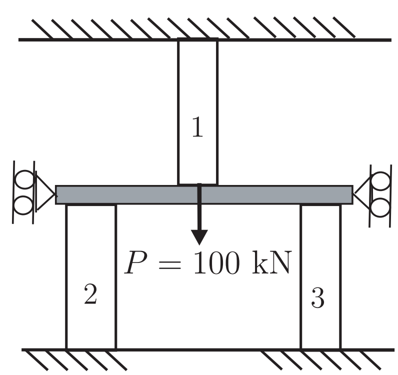
*REQUIRED*: Determine the Young's Modulus of the bar material so that the bar will elongate:

a) No more than 2 inches.
b) No more than 0.2 inches.
c) For and Young's Modulus of $10^6$ psi, find $P(x)$, $u(x)$, and $\sigma_{xx}(x)$.
d) For and Young's Modulus of $10^6$ psi, plot $P(x)$, $u(x)$, and $\sigma_{xx}(x)$.




Bar #1: Aluminum
$E = 70$ GPa
Area $= 0.0002$ m$^2$
Length $= 30$ cm
Bar #3: Steel
$E = 200$ GPa
Area $= 0.0002$ m$^2$
Length $= 20$ cm
Bar #4: Aluminum
$E = 70$ GPa
Area $= 0.0002$ m$^2$
Length $= 50$ cm

Problem 11.17



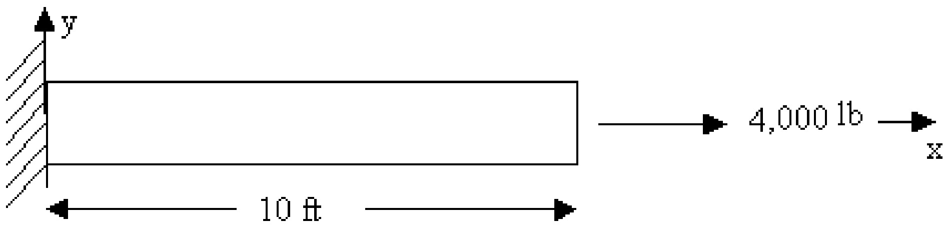

Problem 11.18

11.21 *GIVEN*: A bar with cross-sectional area $A = 5$ in$^2$ is loaded as shown:
*REQUIRED*: For a Young's Modulus of $10^6$ psi:

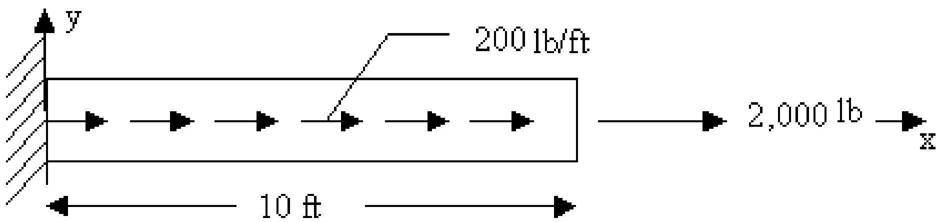
a) Find $P(x)$, $u(x)$, and $\sigma_{xx}(x)$.
b) Plot $P(x)$, $u(x)$, and $\sigma_{xx}(x)$.

11.22 *GIVEN*: A bar with cross-sectional area of $A = 5$ in$^2$ is loaded as shown:
*REQUIRED*: For a Young's Modulus of $10^6$ psi:

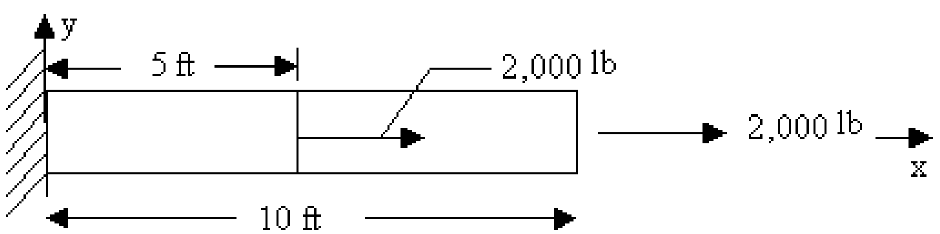
a) Find $P(x)$, $u(x)$, and $\sigma_{xx}(x)$.
b) Plot $P(x)$, $u(x)$, and $\sigma_{xx}(x)$.




Problem 11.20




Problem 11.21




Problem 11.22

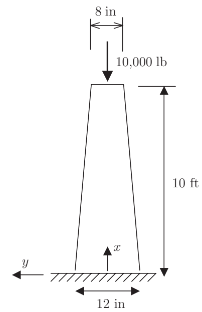
11.23 *GIVEN*: A concrete column support for a bridge as shown to the right:

*ASSUME*:

1) Square cross-section
2) $E = 4.5 \times 10^6$ psi and $\rho = 2242\ \frac{\text{kg}}{\text{m}^3}$

*REQUIRED*: Determine the displacement at:

a) The bottom of the support
b) The middle of the support

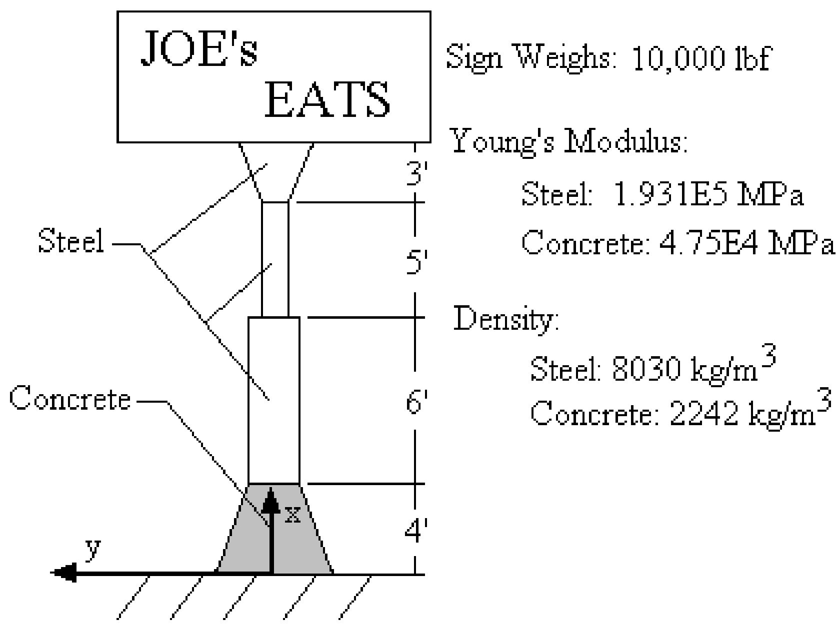



Problem 11.23

c) The top of the support

Useful formula: $\int (a+bx)^n \, dx = \frac{(a+bx)^{n+1}}{(n+1)b}$ for $n \neq -1$

11.24 *GIVEN*: Joe's Restaurant sign as shown below:




Problem 11.24

In this problem, gravity cannot be neglected. *Note: mixed units. Work problem in metric units.*

*ASSUME*: All four bars are cylindrical and:

1) The concrete base has a maximum radius of 2' and a minimum radius of 1'.
2) The steel inverted base has a maximum radius of 1.3' and a minimum radius of 6".

*REQUIRED*:

a) Find $P(x)$, $u_x(x)$, and $\sigma_{xx}(x)$ in the concrete base. $P(x)$ is the internal axial force.

b) Plot $P(x)$, $u_x(x)$, and $\sigma_{xx}(x)$ in the concrete base.