Chapter 13

STRESS AND DEFORMATION ANALYSIS OF LINEAR ELASTIC BEAMS IN BENDING

13.1 Introduction

In Chapters 11 and 12, the analysis of bars subjected to axial and torsional loads was considered. In this chapter, we continue the study of long slender straight geometries but now consider loads which cause bending of the beam, i.e., the loads are transverse to the length of beam and produce transverse displacements and internal shear and moment. We will refer to structural members that exhibit beam bending as beam or frame members. Some examples of frame structures where beam bending is important are shown below. Typically, these transverse loads are due to design loads that the structure must carry along the length of the member (for example, vehicle traffic on a bridge, snow loads on a roof, lift force on an airplane wing) in addition to the weight of the structural member itself. In some cases, the structure may look similar to truss structures that were considered in ENGR 211 as shown below.

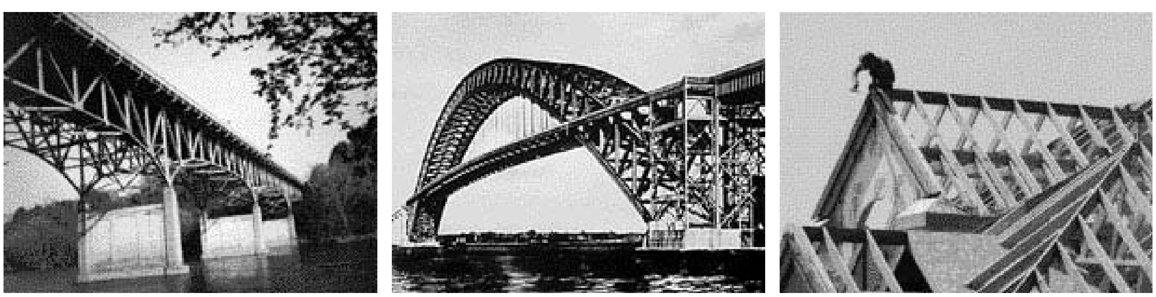


Figure 13.1: Examples of Frame Structures which Exhibit Beam Bending Behavior

However, unlike truss structures which were assumed to be pinned at their joints so that members could carry no shear or bending moment (truss members are two-force members), frame structures and frame members are typically rigidly attached at their end points to supports or other frame members. When one frame member is rigidly attached to another at a joint, the members will transfer axial and shear force as well as a bending moment to each other at the joint. In most structural applications, the shear and bending loads occur in all three coordinate directions and produce deformation in all three coordinate directions. In this chapter, we will consider bending only in a single plane.

Figure 13.2: Cantilever Beams; Manmade and Nature's Own

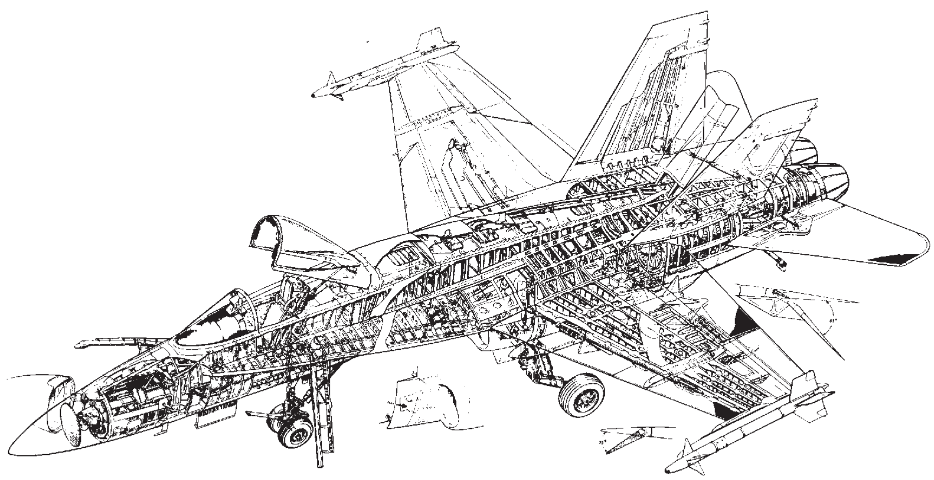


Figure 13.3: Structural Members in a Fighter Aircraft

We will begin our study of beam bending by first considering the bending and shear stress that results from the application of transverse loading, shear and bending moments (so-called pure bending theory). Then we will consider the development of shear and bending moment diagrams (the distribution of shear and bending moment along the length of a beam) and the mathematical relationships between internal shear, bending moment and the applied distributed transverse load. Finally we will address the determination of the transverse displacement that occurs in beam bending.

Note: In the development of a theory to analyze beams under bending, we may take either of two approaches: phenomenological or theoretical (based on solution of the governing mathematical equations). The phenomenological approach means that we observe the phenomenon (usually the kinematics) in the laboratory, postulate appropriate assumptions for certain displacement, strain and/or stress components based on these observations; and develop a theory based on these assumptions in concert with necessary theoretical requirements. The more theoretical approach starts with the basic conservation laws (in this case, linear and angular momentum and possibly energy), the kinematic equations (strain-displacement relations), the appropriate constitutive equations and the appropriate boundary conditions; and solves these in more-or-less exact form. Since the conservation laws are partial differential equations, this requires the solution of a system of partial differential equations. Most of the early developments in beam bending theory by pioneers such Leonard Euler (1707-1783) and Jacob Bernoulli (1654-1705) were for the most part phenomenological. The Euler-Bernoulli theory of bending will be discussed here. Later developments by Lagrange, Navier, Cauchy, Green and Timoshenko (1878-1972), to name a few, followed more theoretical approaches to mechanics. Students interested in further reading on the history of strength of materials (as beam bending theory is often called) may wish to consult Timoshenko's book (*History of Strength of Materials*, S.P. Timoshenko, McGraw-Hill, New York, 1953).

13.2 Bending Stress and Deflection Equation

In this section, we consider the case of *pure bending*; i.e., where only bending stresses exist as a result of applied bending moments. To develop the theory, we will take the phenomenological approach to develop what is called the "***Euler-Bernoulli theory of beam bending***."

Geometry: Consider a long slender straight beam of length L and cross-sectional area A. We assume the beam is prismatic or nearly so. The length dimension is large compared to the dimensions of the cross-section. Schematically, a beam member shown below:

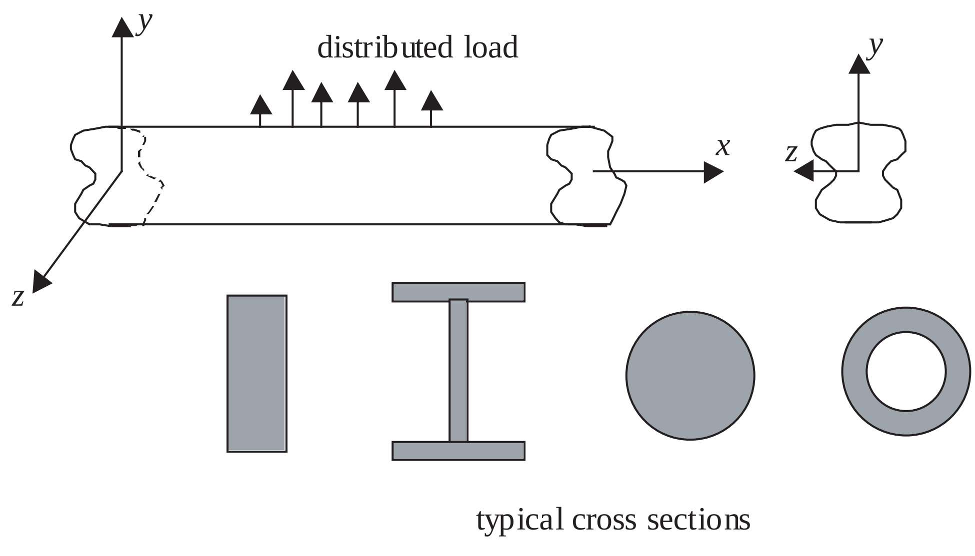


Figure 13.4: Beam Geometry

While the cross-section may be any shape, we will assume that it is *symmetric about the y axis*.

Loading: For our purposes, we will consider shear forces or distributed loads that are applied in the y direction only (on the surface of the beam) and moments about the z-axis. We have consider examples of such loading in ENGR 211 previously and some examples are shown below:

Kinematic Observations: In order to obtain a "feel" for the kinematics (deformation) of a beam subjected to pure bending loads, it is informative to conduct an experiment. Consider a rectangular

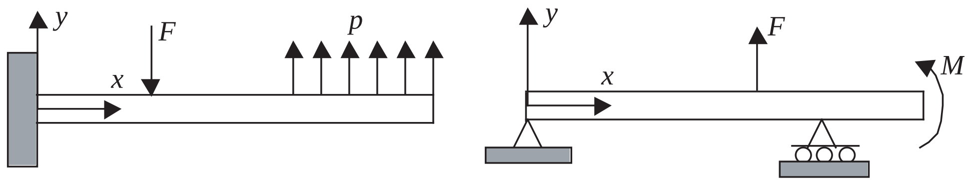


Figure 13.5: Typical Loading on Beam

bar bent by end moments as shown below:

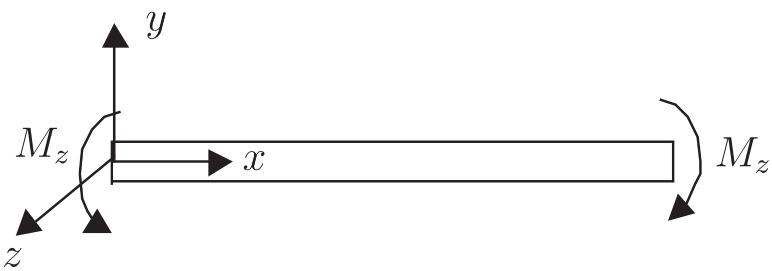


Figure 13.6: Bending about the z-axis by End Moments

The following photograph shows a long beam with a square cross-section. Straight longitudinal lines have been scribed on the beam's surface, which are parallel to the top and bottom surfaces (and thus parallel to a centroidally placed x-axis along the length of the beam). Lines are also scribed around the circumference of the beam so that they are perpendicular to the longitudinals (these circumferential lines form flat planes as shown). The longitudinal and circumferential lines form a square grid on the surface.

The beam is now bent by moments at each end as shown in the lower photograph. After loading, we note that the top line has stretched and the bottom line has shortened (implies that there is strain ε_{xx}). If measured carefully, we see that the longitudinal line at the center has not changed length (implies that $\varepsilon_{xx} = 0$ at $y = 0$). The longitudinal lines now appear to form concentric circular lines. We also note that the vertical lines originally perpendicular to the longitudinal lines remain straight and perpendicular to the longitudinal lines. If measured carefully, we will see that the vertical lines remain approximately the same length (implies $\varepsilon_{yy} = 0$). Each of the vertical lines (as well as the planes they form) has rotated and, if extended downward, they will pass through a common point that forms the center of the concentric longitudinal lines (with some radius ρ). The flat planes originally normal to the longitudinal axis remain essentially flat planes and remain normal to the deformed longitudinal lines. The squares on the surface are now quadrilaterals and each appears to have tension (or compression) stress in the longitudinal direction (since the horizontal lines of a square have changed length). However, in pure bending we make the assumption that. If the x-axis is along the length of beam and the y-axis is normal to the beam, this suggests that we have an axial normal stress σ_{xx} that is tension above the x-axis and compression below the y-axis. The remaining normal stresses σ_{yy} and σ_{zz} will generally be negligible for pure bending about the z-axis. For pure bending, all shear stresses are assumed to be zero. Consequently, for pure bending, the stress matrix reduces to:

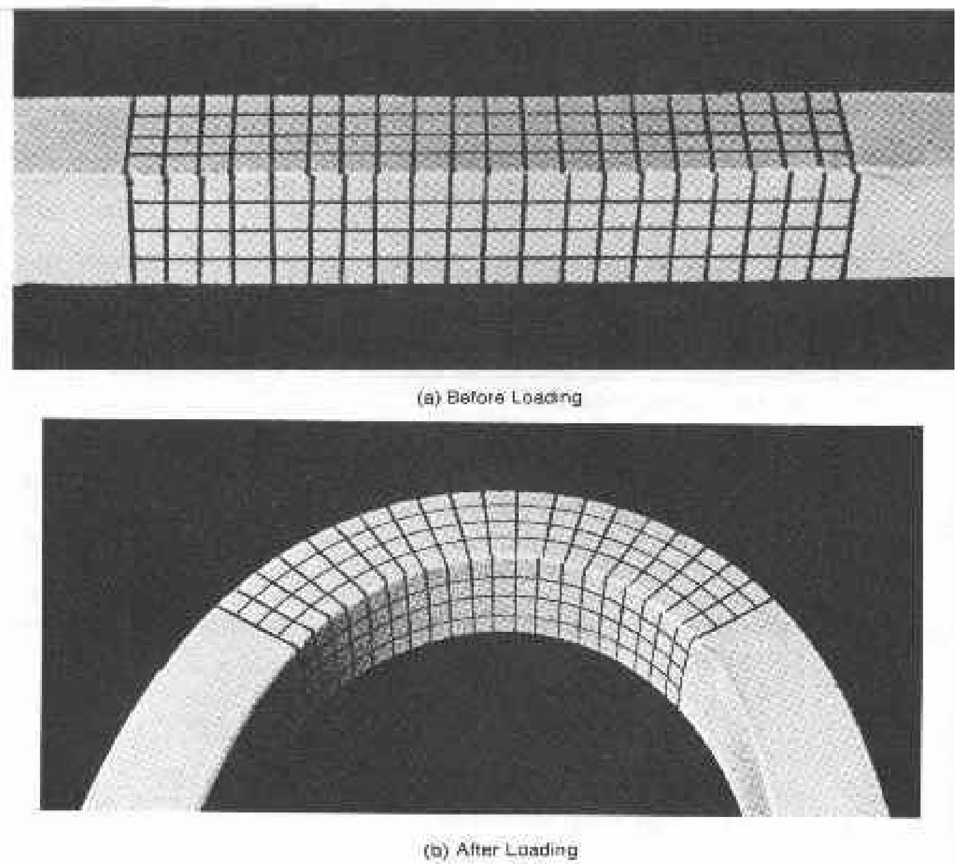


Figure 13.7: Experimental Demonstration of Kinematics in Beam Bending

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{13.1}$$

Later, we will show that when a transverse shear is applied to the structure, shear stresses will not be zero.

Shear and Moment Resultants: Consider a cross-section subjected to internal normal and shear tractions, which can be written equivalently as normal and shear stresses. These tractions may be resolved into axial force, shear and moment resultants. Consider a general distribution of normal and shear stress on a cross-section as shown below and their equivalent resultant forces:

It should be noted that the stress distribution is applied over the cross-sectional area A but is approximately uniform (constant) in the z direction (because there is bending only about the z-axis). In order to define P, V_y and M_z, we require that they provide the same axial and shear force and moment as the stress distribution, i.e., that forces and moments must be equivalent in both cases. Consider the shear and normal stress applied over a small differential area of the cross-section as shown below:

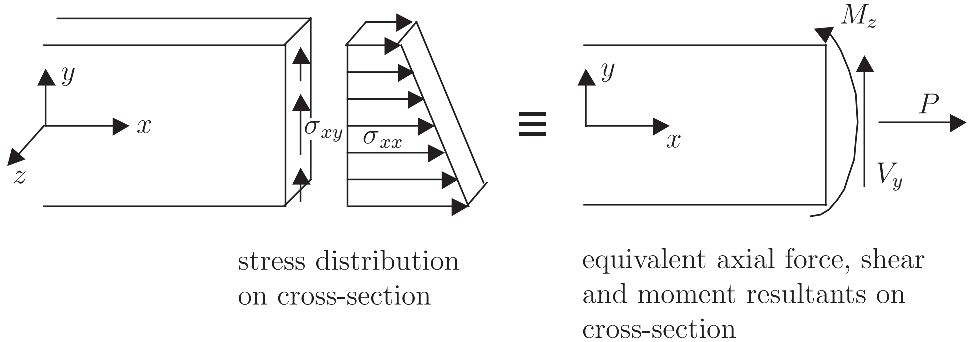


Figure 13.8: Stress Distribution and Equivalent Axial Force, Shear and Moment Resultants on a Cross Section

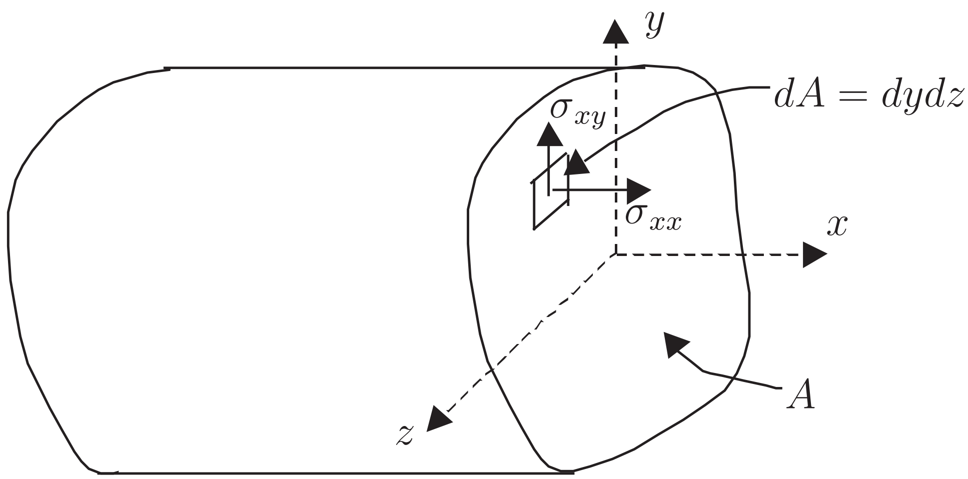


Figure 13.9: Axial and Shear Stress on Beam Cross-Section

The resultant axial force P must be equal to the integral of axial stress σ_{xx} over the cross-section. Likewise, the resultant shear force V_y in the y direction must be equal to the integral of shear stress σ_{xy}, which is also directed in the y direction. The axial stress σ_{xx} (in Figure 13.9) produces a moment about the z-axis that must be equivalent to the moment resultant M_z (in Figure 13.8). Hence, we can write:

$$\begin{aligned} P &= \int_A \sigma_{xx}\, dA \\ V_y &= \int_A \sigma_{xy}\, dA \\ M_z &= -\int_A \sigma_{xx} y\, dA \end{aligned} \tag{13.2}$$

The minus sign is required to be consistent with the positive direction chosen for the moment M_z about the z-axis in Figure 13.8 (note that M_z follows the right-hand rule since the moment vector for M_z would be in the $+z$ direction).

The force and moment resultants P, V_y and M_z are important because they allow us to work with equivalent vector forces on the cross-section instead of dealing with the actual stress distribution. For the pure bending case, we will only need the moment resultant; however, when shear loading is considered, we will need the shear resultant as well.

Constitutive Relations: We assume the material is a homogeneous linear elastic isotropic solid so that the constitutive relations reduce to:

$$\begin{aligned} \sigma_{xx} &= E\varepsilon_{xx} \\ \sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy} = G\gamma_{xy} \end{aligned} \tag{13.3}$$

Kinematic or Strain Displacement Equations: We assume small strain theory and write only the axial strain component for the time being:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \tag{13.4}$$

Kinematic Assumptions: From the previous discussion regarding experimentally observed behavior of beam bending, we draw a sketch showing the beam before and after bending. After bending, the transverse displacement of the centroidal x-axis will be defined by $u_{0y}(x)$ as shown below. The subscript "0" means that u_y is measured at $y = 0$ (i.e., at the centroidal axis position). If we assume that there is no strain in the y direction ($\varepsilon_{yy} = 0$), then u_y is not a function of y. Hence we can write the transverse displacement of any point on the beam in terms of its centroidal value so that

$$u_y(\,x, y\,) = u_{0y}(x) \tag{13.5}$$

The rotation of the beam at any point x is given by the derivative of the transverse displacement u_{0y} with respect to x:

$$\theta(x) = \frac{du_{0y}}{dx} \tag{13.6}$$

Since me make the assumption that a normal to the centroidal axis remains straight and normal, then the normal will also rotate by this same amount θ. For a point "A" located at some position y above the centroidal axis, we note that point A will have moved to the left as shown on the sketch. This displacement in the x direction is the displacement $u_x(\,x, y\,)$ and from the geometry can be written as:

$$u_x(\,x, y\,) = -y\tan^{-1}\theta(x) \approx -y\theta(x) = -y\frac{du_{0y}(x)}{dx} \tag{13.7}$$

Equation (13.7) states that the axial displacement u_x can be written entirely in terms of the transverse displacement of the centroidal axis and that the displacement is linear with transverse position y. Substituting equation (13.7) into (13.4), the axial strain can now be written as

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial\left(-y\frac{du_{0y}(x)}{dx}\right)}{\partial x} = -y\frac{d^2u_{0y}(x)}{dx^2} \tag{13.8}$$

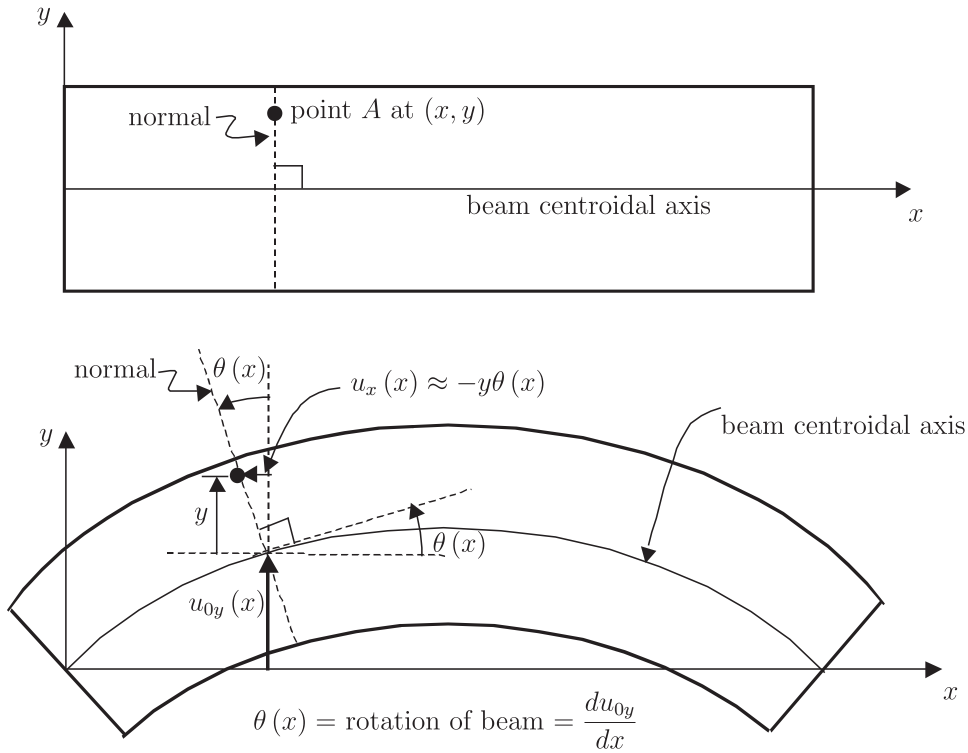


Figure 13.10: Displacement and Rotation in Beam Bending

Note that equation (13.8) satisfies the assumptions that the axial strain at $y = 0$ (the centroidal axis) is zero. Because the strain is zero along the x-axis passing through the centroid, it is sometimes referred to as the *neutral axis*. We can now rewrite the internal bending moment in terms of displacements by substituting the strain-displacement equation (13.8) into the stress-strain equation (13.3) and that result into the moment equation (13.2) to obtain

$$M_z = -\int_A \sigma_{xx} y \, dA = -\int_A E\varepsilon_{xx} y \, dA = -\int_A E\left(-y\frac{d^2u_{0y}(x)}{dx^2}\right) y \, dA \tag{13.9}$$

Note that we must integrate over the cross-section, A which lies in the y-z plane. We now assume that Young's modulus E is a constant over the cross-section, i.e., $E = E(x)$ and thus E may be taken outside the integral. Since $u_{0y}(x)$ is not a function of y or z, it may also be taken outside the integral. Hence, we write:

$$M_z = E\frac{d^2u_{0y}(x)}{dx^2}\int_A y^2 \, dA \tag{13.10}$$

The integral term is a geometrical property of the cross-section and can be easily integrated:

$$I_{zz} = \int_A y^2 \, dA \tag{13.11}$$

where I_{zz} is called the *moment of inertia* of the cross-section about the z axis. As stated previously, we have assumed that *the cross-section is symmetric about the y axis.* If the cross-section is not symmetric about the y axis, a transverse load may produce twisting of the cross-section that we have not considered here. Note that bending is occurring about the z axis since bending moments are about the z axis. With this definition of the moment of inertia, equation (13.10) now becomes

$$M_z = EI_{zz}\frac{d^2u_{0y}(x)}{dx^2}$$

or

$$\frac{d^2u_{0y}(x)}{dx^2} = \frac{M_z}{EI_{zz}} \tag{13.12}$$

Equation (13.12) is an ordinary second order differential equation that defines the transverse displacement in terms of the bending moment. Since the bending moment M_z will in general be a function of x, it will be necessary to determine this moment expression $M_z(x)$ before equation (13.12) can be integrated.

The stress may now be written in terms of the bending moment by substituting equation (13.12) into the strain equation (13.8) and the result substituted into the stress equation (13.3) to obtain:

$$\sigma_{xx} = E\varepsilon_{xx} = E\left(-y\frac{d^2u_{0y}(x)}{dx^2}\right) = E\left(-y\frac{M_z}{EI_{zz}}\right)$$

or

$$\sigma_{xx} = -\frac{M_z y}{I_{zz}} \tag{13.13}$$

Note that $\sigma_{xx} = \sigma_{xx}(x, y)$. Equation (13.13) shows that for any position x along the length of the beam, the bending stress varies linearly with y (i.e., linearly from top to bottom surface of the beam) and is zero at the centroidal axis. The linear variation of bending stress through the cross-section is shown below.

Since the bending moment $M_z = M_z(x)$, then the stress also varies with position along the length of the beam. This moment distribution may be determined by using free-body diagrams to be discussed below.

13.3 Review of Centroids and Moments of Inertia

Before we consider the application of beam bending theory developed up to this point, we note that it is necessary to establish the location of the centroid for the cross section as well as the moment of inertia about the bending axis. Consequently, we review these topics before going further.

Consider a composite cross-section consisting of two separate areas as shown below. We wish to determine the location of the centroid of the cross-section.

Relative to some y'-z' reference axes, define the centroid of the composite area to be $\bar{y}$, and the centroid of each sub-area to be $\bar{y}_i$. We require the "first moment of the area" about the z' axis in terms of the discrete values $\bar{y}A$ to be equal to the integral value $\int_A y'\,dA$ so that we write:

$$\bar{y}A = \int_A y'\,dA \rightarrow \bar{y}\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \bar{y}_i A_i$$

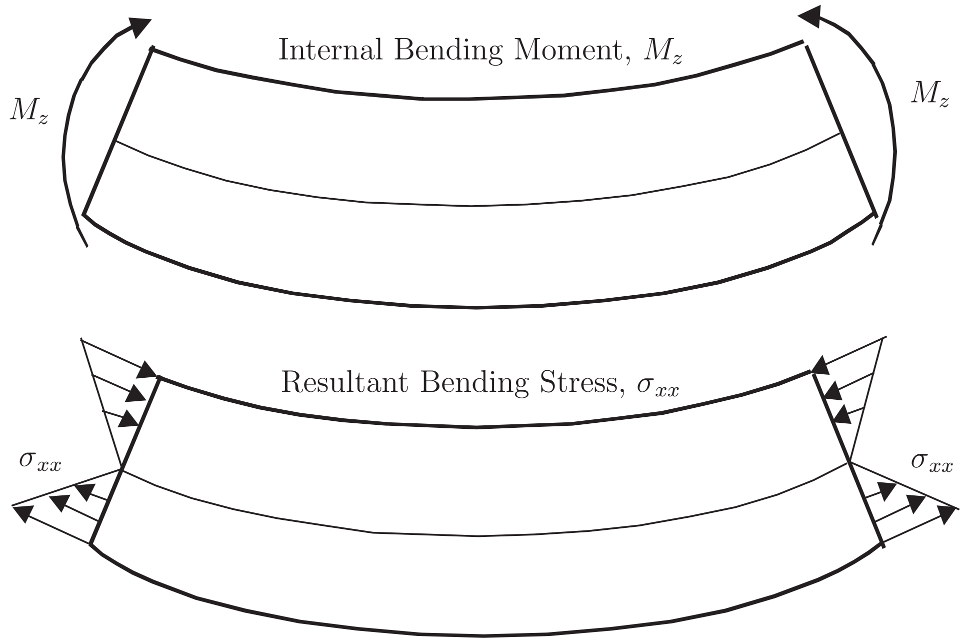


Figure 13.11: Internal Bending Stress Distribution for Beam Bending

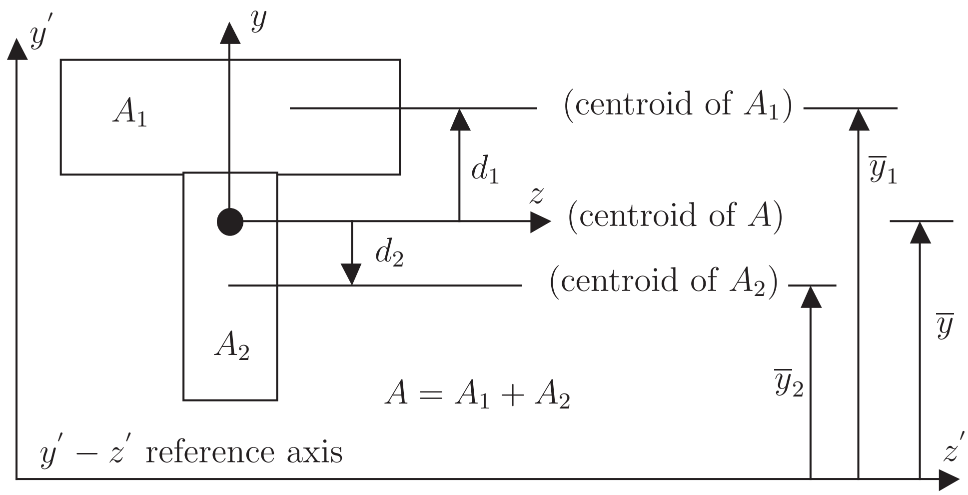


Figure 13.12: Sketch for Determining Location of Centroid

Solving for $\bar{y}$ gives:

$$\bar{y} = \frac{\int_A y\, dA}{A} = \frac{\sum_{i=1}^{n} \bar{y}_i A_i}{\sum_{i=1}^{n} A_i} \tag{13.14}$$

The moment of inertia about the bending axis was given by equation (13.11). We can also develop the parallel axis theorem (also called the transfer theorem) for moments of inertia. Suppose that we know the moment of inertia of an area about its centroid and wish to determine the moment of inertia about some other parallel axis. Consider the following sketch of a rectangular area (Note a rectangular area is shown for simplicity; however, the area can be any shape.):

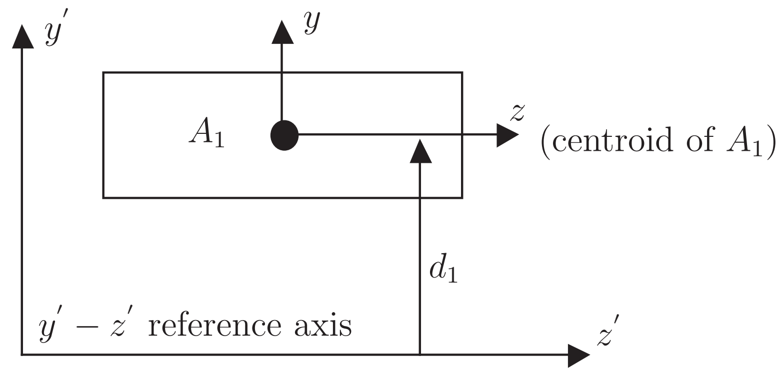


Figure 13.13: Sketch for Developing Parallel Axis Theorem

Referring to the figure above, consider area A_1. The moment of inertia of this area about the z' axis is defined by

$$I_{z'z'} = \int\limits_{A_1} (y')^2 \, dA$$

The y and y' coordinates are related by the transformation $y' = y + d_1$. Substituting this into the above gives

$$I_{z'z'} = \int\limits_{A_1} (y')^2 \, dA = \int\limits_{A_1} (y + d_1)^2 \, dA = \int\limits_{A_1} y^2 \, dA + \int\limits_{A_1} 2yd_1 \, dA + \int\limits_{A_1} (d_1)^2 \, dA$$

The first term on the right is the moment of inertia about the z-axis passing through the centroid of area A_1: $\bar{I}_{zz_1} \equiv \int\limits_{A_1} y^2 \, dA$

The second term can be written as $2d_1 \int\limits_{A_1} y \, dA$ since d_1 is a constant. However, the term $\int\limits_{A_1} y \, dA = 0$ since the y-z axes is located at the centroid of A_1. The last term is simply $(d_1)^2 A_1$. Consequently we can write

$$I_{z'z'} = \bar{I}_{zz_1} + A_1 d_1^2 \tag{13.15}$$

This last result is called the parallel axis theorem or transfer theorem. It allows one to determine the moment of inertia about a parallel axes (z') in terms of moment of inertia about the centroidal axis (z) and the distance between z and z' (d_1). Now consider a composite body made of n sub-areas A_i such as that shown below:

The parallel axis theorem for a single area (13.15) can be generalized to obtain the following expression for determining the moment of inertia about the centroidal axis of the composite body:

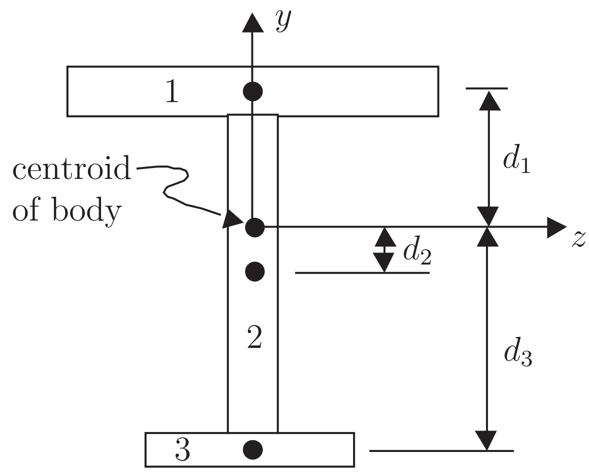


Figure 13.14: Parallel Axis Theorem for Composite Body

$$I_{zz} = \int_A y^2\, dA = \sum_{i=1}^{n} (\bar{I}_{zz_i} + A_i d_i^2) \tag{13.16}$$

where

$$\begin{aligned} I_{zz} &= \text{moment of inertia of body about its centroidal axis} \\ \bar{I}_{zz_i} &= \text{moment of inertia of area i about its centroidal axis} \\ d_i &= \text{distance between centroid of area i and centroidal axis of body} \\ A_i &= \text{area } i \end{aligned}$$

Example 13-1

Consider a cross-section of dimensions b and h as shown below. Determine the location of the centroid and the moment of inertia about the z axis (located at the centroid).

By inspection the centroid is located at the center of the cross-section, i.e., $\frac{h}{2}$ from the top and $\frac{b}{2}$ from the left edge.

$$I_{zz} = \int_A y^2\, dA = \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 b\, dy = b \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2\, dy = \frac{bh^3}{12}$$

In a similar fashion, the moment inertia about the y axis is given by:

$$I_{yy} = \int_A z^2\, dA = \int_{-\frac{b}{2}}^{\frac{b}{2}} z^2 h\, dz = h \int_{-\frac{b}{2}}^{\frac{b}{2}} z^2\, dz = \frac{hb^3}{12}$$

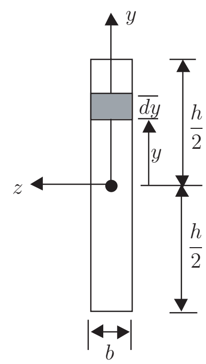


Figure 13.15:

We note in passing that for a *circular cross-section* of diameter D, we find

$$I_{yy} = I_{zz} = \frac{\pi D^4}{64} \text{ and } J = I_{yy} + I_{zz} = \frac{\pi D^4}{32}$$

Example 13-2

Determine the centroid and moment of inertia of the following composite body.

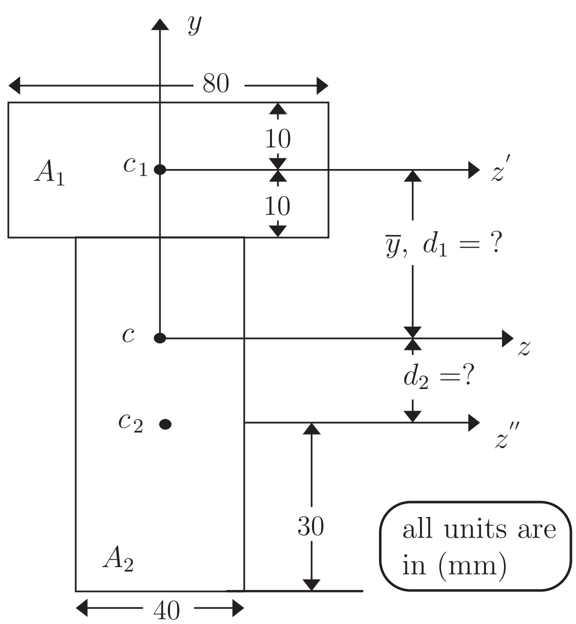


Figure 13.16:

The centroid of the composite body is labeled "c". The centroid of area 1 and 2 is labeled c_1 and c_2, respectively.

Solution

By inspection, the horizontal location of the centroid of the composite body is 20 cm to the right of the left edge of the lower area. To determine the vertical location, we use $\bar{y} = \frac{\sum_i A_i \bar{y}_i}{\sum_i A_i}$. Choose a reference z' axis to located at the centroid of the upper area. Then we write:

$$\bar{y} = \frac{\sum_i \bar{y}_i A_i}{\sum_i A_i} = \frac{0(80 \times 20) + (-40)(40 \times 60)}{(80 \times 20) + (40 \times 60)} = -24 \text{ mm}$$

Hence the vertical position of the centroid for the composite body is located 24 mm below the centroid of area 1, or 34 mm below the top of the body.

Now, we determine the moment of inertia about the centroidal axis of the composite body using the parallel axis theorem. Knowing the location of the centroid, we know that $d_1 = 24$ mm, $d_2 = -16$ mm. We will determine the moment of inertia of each area separately and then sum them.

$$A_1 \rightarrow \begin{array}{l} I^{(1)}_{z'z'} = \frac{1}{12}bh^3 = \frac{1}{12}(80 \times 20^3) = 53.3 \times 10^3 \qquad [\text{mm}^4] \\ I^{(1)}_{zz} = I^{(1)}_{z'z'} + A_1 d_1^2 = 53,300 + (80 \times 20)(24)^2 = 975,000 \qquad [\text{mm}^4] \end{array}$$

$$A_2 \rightarrow \begin{array}{l} I^{(2)}_{z''z''} = \frac{1}{12}40(60)^3 = 720,000 \qquad [\text{mm}^4] \\ I^{(2)}_{zz} = I^{(2)}_{z''z''} + A_2 d_2^2 = 720,000 + (40 \times 60)(-16)^2 = 1,334,000 \qquad [\text{mm}^4] \end{array}$$

$$\text{Composite Area}\} \implies \quad I_{zz} = I^{(1)}_{zz} + I^{(2)}_{zz} = 975,000 + 1,334,000 = 2.31 \times 10^6 \qquad [\text{mm}^4]$$

Example 13-3

Consider a beam subjected to pure bending with rectangular cross-section of width "b" and height "a" as shown below. At a certain location x of the beam, assume that the bending moment is given by $M_z = 7,000$ N m. Determine the maximum bending stress.

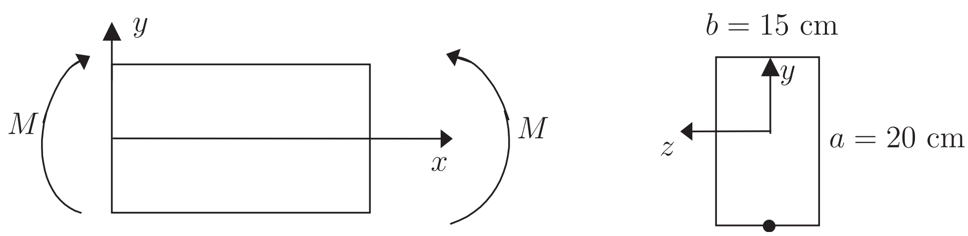


Figure 13.17:

The moment of inertia is given by:

$$I_{zz} = \frac{1}{12}ba^3 = \frac{1}{12}\left(15 \times 10^{-2} \text{ m}\right)\left(20 \times 10^{-2} \text{ m}\right)^3 = 10^{-4} \text{ m}^4$$

The axial stress is given by

$$\sigma_{xx} = -\frac{M_z y}{I_{zz}} = -\frac{(7,000 \text{ N m})y}{10^{-4} \text{ m}^4}$$

The stress is a maximum at either the upper or lower surface of the beam. At the lower surface ($y = -10$ cm), the stress is given by

$$\sigma_{xx} = -\frac{M_z y}{I_{zz}} = -\frac{(7,000 \text{ N m})(-0.1 \text{ m})}{10^{-4} \text{ m}^4} = 7 \times 10^6 \ \frac{\text{N}}{\text{m}^2} = 7 \text{ MPa}$$

Note that the stress is positive indicating a tensile stress at the lower surface. Given the direction of the bending moment, we would expect tension on the lower surface and compression on the upper surface.

13.4 Shear and Bending Moment Relationships

In order to determine the axial stress σ_{xx} at any point x along the length of the beam from equation (13.13), or to determine the deflection of the beam $u_{0y}(x)$ from equation (13.12), it is necessary that we have the expression for the bending moment $M_z(x)$. In this section, we develop the methods for determining the bending moment distribution $M_z(x)$ but also the shear distribution $V_y(x)$ in terms of the applied loads.

Consider first a beam that has a distributed load p_y acting in the y direction along its top edge as shown below. It is assumed that p_y has units for $\frac{\text{force}}{\text{length}}$ and is a function of x.

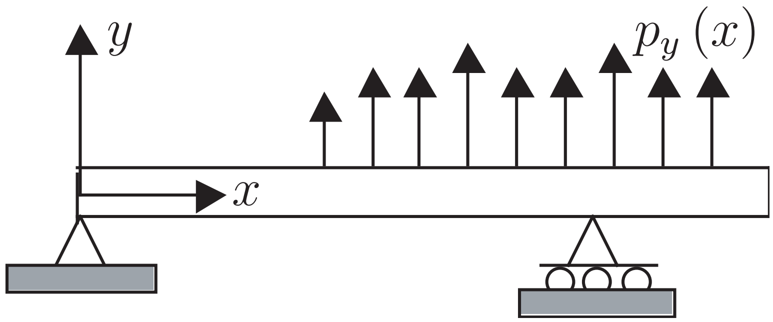


Figure 13.18: Beam with Distributed Load Applied to Top Surface

Note that the distributed load would normally be applied as a pressure on the top surface but the pressure can be multiplied by the width of the top surface to obtain p_y which would have units of force per length in the x direction. Note also that the load may be applied either to the upper or lower surface of the beam since this results in identical bending moments. Now consider a free-body of the beam of length Δx taken at some position x. At this location x, we assume the distributed load $p_y(x)$is applied. The distributed load will cause axial stress σ_{xx} and shear stress σ_{xy} within the beam but these may be replace by the equivalent resultants P, V_y and M_z as was shown above in equation (13.2). Consequently, we have the free-body diagram shown below:

Note the following **sign convention for shear and bending moment**: on the right face, the shear is positive in the $+y$ direction, and the bending moment is positive in the counter clockwise direction (follows right hand rule for moments). On the left face, the positive directions are reversed in order to satisfy conservation of linear and angular momentum. The positive direction of p_y is in the $+y$ direction. Applying conservation of linear momentum in the x direction gives:

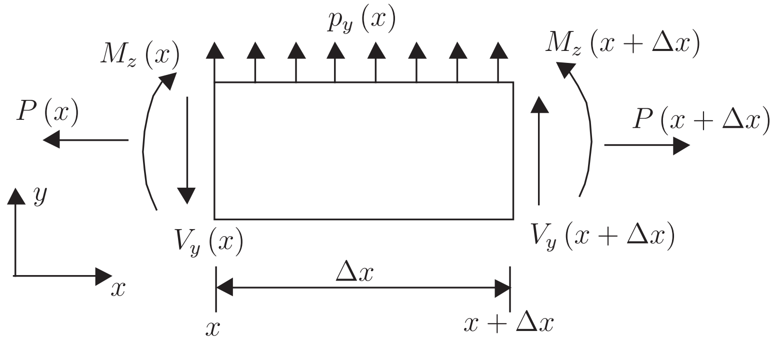


Figure 13.19: Free-Body of Differential Length of Beam

$$0 = P(x + \Delta x) - P(x)$$

Dividing by Δx and taking the limit as $\Delta x \to 0$ gives

$$\frac{dP}{dx} = 0 \tag{13.17}$$

Equilibrium in the y direction gives

$$0 = V_y(x + \Delta x) - V_y(x) + p_y \Delta x$$

Dividing by Δx and taking the limit as $\Delta x \to 0$ gives

$$\frac{dV_y}{dx} = -p_y \tag{13.18}$$

Applying conservation of angular momentum (z-component) about the center of differential element gives:

$$0 = M_z(x + \Delta x) - M_z(x) + V(x + \Delta x)\left(\frac{\Delta x}{2}\right) + V(x)\left(\frac{\Delta x}{2}\right) + \int_{x+\frac{\Delta x}{2}}^{x+\Delta x} p_y\,dx - \int_{x}^{x+\frac{\Delta x}{2}} p_y\,dx$$

Dividing by Δx and taking the limit as $\Delta x \to 0$ gives

$$\frac{dM_z}{dx} = -V_y \tag{13.19}$$

Note that the distributed load terms produce equal and opposite moments about the center as $\Delta x \to 0$ and they cancel out.

Thus, we have the following three equilibrium equations relating P, V_y and M_z to the applied distributed load p_y:

$$\begin{aligned}\frac{dP}{dx} &= 0\\ \frac{dV_y}{dx} &= -p_y\\ \frac{dM_z}{dx} &= -V_y\end{aligned} \tag{13.20}$$

Equations (13.18) and (13.19) may be combined to obtain:

$$\frac{d^2M_z}{dx^2} = -\frac{dV_y}{dx} = p_y \tag{13.21}$$

Equations (13.20) indicate that the bending moment is always one order higher polynomial in x than the shear; i.e., if the shear is a constant, then the bending moment is linear in x and so forth.

Recalling the definition of a maximum from calculus, we see that the shear-moment relation indicates that

- M_z is a maximum (or minimum) at the point where $V_y = 0$.

  Also, from the shear-load relationship, we see that

- the slope of the V_y curve at point x is equal to the value $-p_y$ at point x, and
- the slope of the M_z curve at point x is equal to the value $-V_y$ at point x.

Equations (13.20) may be integrated to obtain the shear and bending moment as a function of x. Integrating each equation from some point x_0 to an arbitrary point x provides the following results:

$$\int_{x_0}^{x} dV_y = -\int_{x_0}^{x} p_y\,dx \qquad \text{or} \qquad V_y(x) = V_y(x_0) - \int_{x_0}^{x} p_y\,dx \tag{13.22}$$

Similarly, for the bending moment equation;

$$M_z(x) = M_z(x_0) - \int_{x_0}^{x} V_y\,dx \tag{13.23}$$

Equation (13.23) states that if the bending moment is known at location x_0, then the bending moment at any point x can be obtained by integrating the shear expression from x_0 to x. If V_y is plotted against x, the integral $\int_{x_0}^{x} V_y\,dx$ represents the area under the shear curve from x_0 to x. This suggest a graphical statement for equations (13.22) and (13.23) as

$$\begin{aligned}V_y(x) &= V_y(x_0) - \text{area under the } p_y \text{ curve from } x_0 \text{ to } x\\ M_z(x) &= M_z(x_0) - \text{area under the } V_y \text{ curve from } x_0 \text{ to } x\end{aligned} \tag{13.24}$$

Example 13-4

Consider a simply supported beam of length $L = 10$ ft with a uniformly distributed normal load of 50 $\frac{\text{lb}}{\text{ft}}$ and a concentrated load of 200 lb as shown below.

Make a free-body of the entire structure by removing the supports and placing the appropriate reactions at the support points. At the pin (left end) we have both x and y reactions; at the roller support, we have only a y (vertical) reaction.

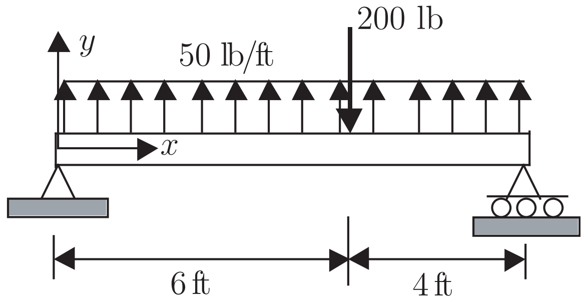


Figure 13.20:

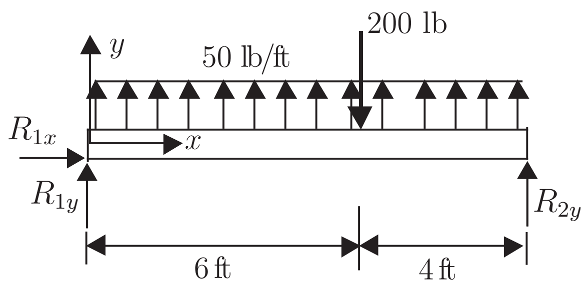


Figure 13.21:

Determine the reactions first. Apply conservation of linear and angular momentum to obtain the following three *equilibrium equations*:

$$
\begin{aligned}
0 &= \sum F_x = R_{1_x} \\
0 &= \sum F_y = R_{1_y} + 500\ \frac{\text{lb}}{\text{ft}}(10\text{ ft}) - 200\text{ lb} + R_{2_y} \\
0 &= \sum M_{\text{left end}} = 500\ \frac{\text{lb}}{\text{ft}}(10\text{ ft})(5\text{ ft}) - 200\text{ lb}(6\text{ ft}) + (10\text{ ft})R_{2_y}
\end{aligned}
$$

Solving for the reactions, we obtain

$$R_{1_x} = 0, \quad R_{1_y} = -170\text{ lb}, \quad R_{2_y} = -130\text{ lb}$$

which can shown on a free-body of the structure:

In constructing shear and moment diagrams, In evaluating the shear and moment, and the constants of integration, **it is imperative to keep in mind the sign convention for shear and bending moment that was established earlier**:

Note that we have the following values of shear and bending moments at the end points:

$$
\begin{aligned}
x = 0: &\qquad V_y(0) = 170\text{ lb}, \quad M_z(0) = 0 \\
x = 10\text{ ft}: &\qquad V_y(10) = -130\text{ lb}, \quad M_z(0) = 0
\end{aligned}
$$

In this example, we will consider three methods to obtain the shear and moment diagrams:

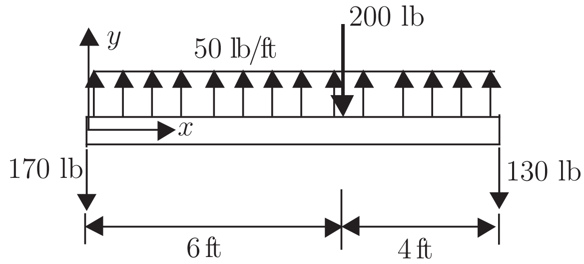


Figure 13.22:

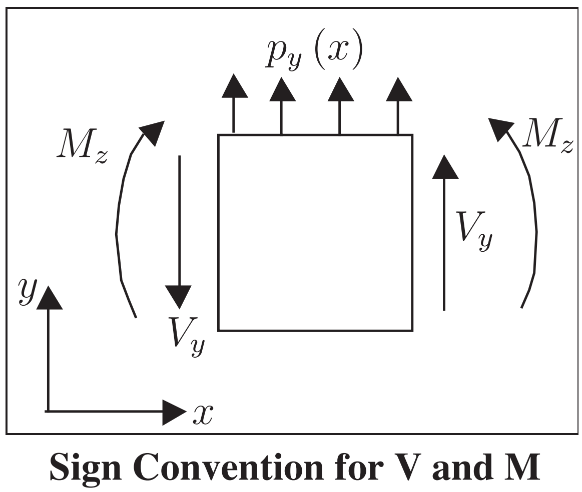


Figure 13.23:

- Free-body diagram method,
- integration method and
- graphical method.

As we will see, each method offers some advantageous but the method of choice will often depend on the problem. For example, when distributed loads are applied to the beam, the integration method is usually preferred.

Free-Body Diagram Method to Obtain V & M

In order to obtain the shear diagram for the complete beam, it will be necessary to consider two free-body diagrams; one for the segment to left of the concentrated force and one to the right of the concentrated force. This is because there will be discontinuity in $V_y(x)$ at $x = 6$ ft where the concentrated shear is applied. First, consider a free-body obtained by cutting *at any point* x in the range $6 < x \leq 10$ (to the right of the concentrated shear load):

$$6 < x \leq 10 \text{ ft}$$

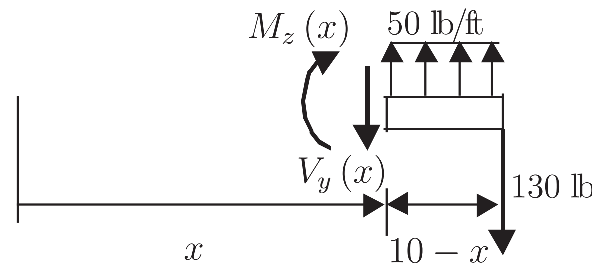


Figure 13.24:

Apply conservation of linear and angular momentum to the free body above to obtain the following equilibrium equations:

$$\begin{aligned}
0 &= \sum F_y = -V_y(x) + 50\,\frac{\text{lb}}{\text{ft}}(10-x)\text{ ft} - 130\text{ lb}\\
0 &= \sum M_{\text{left end}} = -M_z(x) + 50\,\frac{\text{lb}}{\text{ft}}(10-x)\text{ ft}\frac{10-x}{2}\text{ ft} - 130\text{ lb}(10-x)\text{ ft}
\end{aligned}$$

Thus we have

$$\left.\begin{array}{l}
V_y(x) = 370 - 50x \text{ (lb)}\\
M_z(x) = 50\frac{(10-x)^2}{2} - 130(10-x) = 1,200 - 370x + 25x^2 \text{ (ft - lb)}
\end{array}\right\} 6 < x \le 10 \text{ ft}$$

Important note: The shear equation above does not apply at the point $x = 6$ because the shear is discontinuous at $x = 6$ due to the 200 lb shear load at $x = 6$.

$0 \le x < 6$ ft

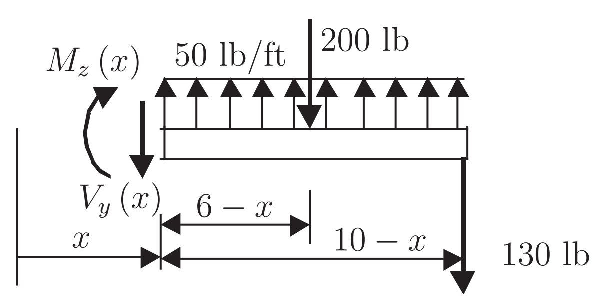


Figure 13.25:

Apply conservation of linear and angular momentum to the free body above to obtain the equilibrium equations:

$$
\begin{aligned}
0 &= \sum F_y = -V_y(x) + 50\,\frac{\text{lb}}{\text{ft}}(10-x)\text{ ft} - 200\text{ lb} - 130\text{ lb} \\
0 &= \sum M_{\text{left end}} = -M_z(x) + 50\,\frac{\text{lb}}{\text{ft}}(10-x)\text{ ft}\frac{10-x}{2}\text{ ft} - 200\text{ lb}(6-x)\text{ ft} - 130\text{ lb}(10-x)\text{ ft}
\end{aligned}
$$

Thus we have

$$
\left.\begin{array}{l} V_y(x) = 170 - 50x \text{ (lb)} \\ M_z(x) = -170x + 25x^2 \text{ (ft - lb)} \end{array}\right\} 0 \le x < 6 \text{ ft}
$$

Important note: The shear equation above does not apply at the point $x = 6$ ft because the shear is discontinuous at $x = 6$ ft due to the 200 lb shear load at $x = 6$ ft.

We can now plot the above equations to obtain the shear and moment distribution.

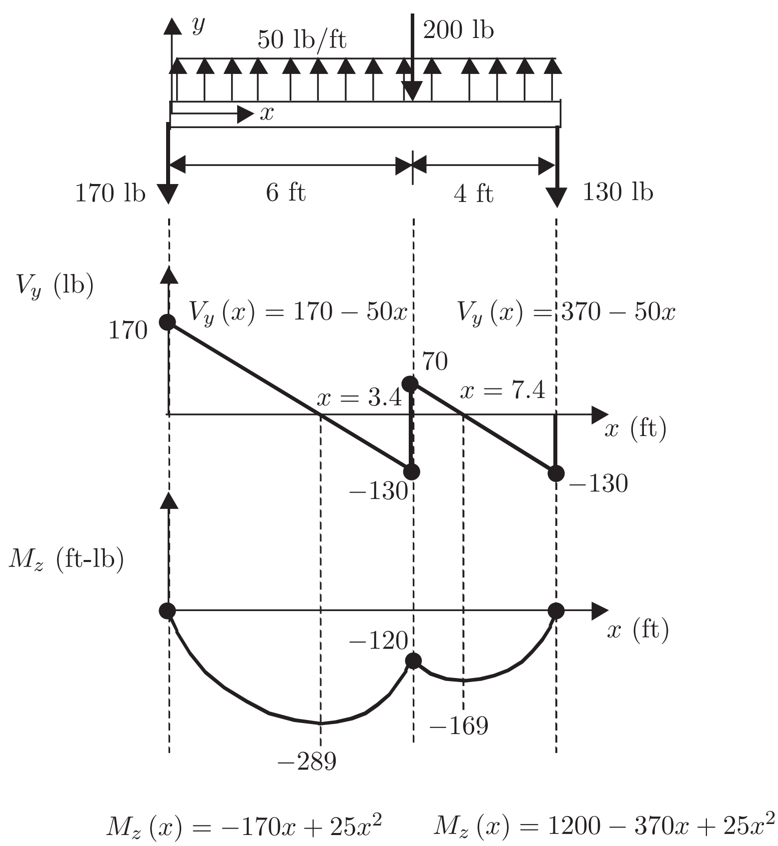


Figure 13.26:

Recall the relationships between moment, shear and load: $\frac{dM_z}{dx} = -V_z$ and $\frac{dV_y}{dx} = -p_y$. Recalling the definition of a maximum from calculus, we see that the shear-moment relation indicates that M_z is a maximum (or minimum) at the point where $V_y = 0$. Also, from the shear-load relationship, the slope of the V_y curve at point x is equal to the value $(-p_y)$ at point x, and the slope of the M_z curve at point x is equal to the value $(-V_y)$ at point x.

For the left section of the beam, we set $V_y(x) = 0 = 170 - 50x$ (lb) and solve for x to obtain $x = 3.4$ ft as the location of the maximum moment. Substituting $x = 3.4$ into the moment equation gives $M_z(3.4) = -170(3.4) + 25(3.4)^2 = -289$ ft - lb. Similarly, for the right portion of the beam (to right of 200 lb load), we obtain $V_y(x) = 0$ at $x = 7.4$ ft and $M_z(7.4) = -169$ ft - lb.

Integration Method to Obtain V & M

In the above, we used conservation of linear and angular momentum together with free-body diagrams to obtain the shear and moment distribution. We can also use the integration method by integrating equations (13.18) and (13.19). Using the integration method, we must start at some point where the shear and bending moment are known. Lets start at $x = 0$ where $V_y(0) = 170$, $M_z(0) = 0$.

$0 \leq x < 6$ ft

We know that $V_y(0) = 170$ lb, $M_z(0) = 0$.
First integrate $\frac{dV_y}{dx} = -p_y$ from 0 to x $(0 \leq x < 6)$ to obtain

$$V_y(x) = V_y(0) - \int_0^x p_y \, dx = 170 - \int_0^x (+50) \, dx = 170 - 50x \text{ (lb)} \quad (0 \leq x < 6 \text{ ft})$$

Important note: The shear equation above does not apply at the point $x = 6$ because the shear is discontinuous at $x = 6$ ft due to the 200 lb shear load at $x = 6$ ft.
Now integrate $\frac{dM_z}{dx} = -V_y$ from 0 to x $(0 \leq x < 6)$ to obtain

$$M_z(x) = M_z(0) - \int_0^x V_y \, dx = 0 - \int_0^x (170 - 50x) \, dx = 170x + 25x^2 \text{ (ft - lb)} \quad (0 \leq x < 6 \text{ ft})$$

$6 < x \leq 10$ ft

To complete the solution, we can integrate either from 6 to x $(x > 6)$ or from 10 to x. It is easier to integrate from 10 to x because we know the starting values of shear and moment at $x = 10$: $V_y(10) = -130$ lb, $M_z(0) = 0$.
As a *first approach*, integrate $\frac{dV_y}{dx} = -p_y$ from 10 to x $(6 < x \leq 10)$ to obtain

$$V_y(x) = V_y(10) - \int_{10}^x p_y \, dx = -130 - \int_{10}^x (+50) \, dx = 370 - 50x \text{ (lb)} \quad (0 \leq x < 6 \text{ ft})$$

Important note: The shear equation above does not apply at the point $x = 6$ because the shear is discontinuous at $x = 6$ due to the 200 lb shear load at $x = 6$.
Now integrate $\frac{dM_z}{dx} = -V_y$ from 10 to x $(6 < x \leq 10)$ to obtain

$$M_z(x) = M_z(10) - \int_{10}^x V_y \, dx = 0 - \int_{10}^x (370 - 50x) \, dx = 1,200 - 370x + 25x^2 \text{ (ft - lb)} \quad (0 \leq x < 6 \text{ ft})$$

If we want take the *second approach* and integrate from 6 to x, we need the shear and moment at $x = 6^+$ (the + means just to the right of 6). This is somewhat tricky because of the 200 lb shear applied at $x = 6$ which creates a discontinuity in the shear at $x = 6$. The correct value of shear to start with would be the value of shear at $x = 6^-$ obtained from the $(0 \leq x < 6)$ solution "plus" the shear load at $x = 6$: $V_y(6) = [170 - 50(6)] + 200 = 70$.

First integrate $\frac{dV_y}{dx} = -p_y$ from 6^+ to x $(6 < x \leq 10)$ to obtain

$$V_y(x) = V_y(6^+) - \int_6^x p_y\,dx = 70 - \int_6^x (+50)\,dx = 70 - 50(x-6) = 370 - 50x \text{ (lb)} \quad (6 < x \leq 10 \text{ ft})$$

Important note: The shear equation above does not apply at the point $x = 6$ because the shear is discontinuous at $x = 6$ due to the 200 lb shear load at $x = 6$.

From the solution for $0 \leq x < 6$, $M_z(6) = 170(6) + 25(6)^2 = -120$ ft - lb.

Now integrate $\frac{dM_z}{dx} = -V_y$ from 6^+ to x $(6 < x \leq 10)$ to obtain

$$M_z(x) = M_z(6) - \int_6^x V_y\,dx = -120 - \int_6^x (370 - 50x)\,dx = 1,200 - 370x + 25x^2 \text{ (ft - lb)} \quad (6 < x \leq 10 \text{ ft})$$

Comparing results for the two different approaches for integrating in the range $(6 < x \leq 10)$, we note that the results are identical.

The shear and moment diagrams may now be sketched as before. See the sketch above.

Graphical Method to Obtain V & M

Recall that the shear and moment are related by

$$\begin{aligned} \frac{dV_y}{dx} &= -p_y \\ \frac{dM_z}{dx} &= -V_y \end{aligned}$$

Integrating each equation from some point x_0 to an arbitrary point x provides the following results:

$$\begin{aligned} V_y(x) &= V_y(x_0) - \int_{x_0}^x p_y\,dx \\ M_z(x) &= M_z(x_0) - \int_{x_0}^x V_y\,dx \end{aligned}$$

If V_y is plotted against x, the integral $\int_{x_0}^x V_y\,dx$ represents the area under the shear curve from x_0 to x. This suggest a graphical statement for the equations as

$$\begin{aligned} V_y(x) &= V_y(x_0) - \text{area under the } p_y \text{ curve from } x_0 \text{ to } x \\ M_z(x) &= M_z(x_0) - \text{area under the } V_y \text{ curve from } x_0 \text{ to } x \end{aligned}$$

Lets use this method to determine the value at $x = 6$ ft. We already know that at $x = 0$, $V_y(0) = 170$ lb and $M_z(0) = 0$. The p_y curve is given by:

$$\begin{aligned} V_y(6) &= V_y(0) - \text{area under the } p_y \text{ curve from 0 to 6} \\ &= 170 \text{ lb} - \left[\left(50\ \frac{\text{lb}}{\text{ft}}\right)(6 \text{ ft})\right] = -130 \text{ lb} \end{aligned}$$

This gives us two points on the V_y curve from $x = 0$ to $x = 6$:

Recall that $\frac{dV_y}{dx} = -p_y$. Since the load p_y is a constant from $x = 0$ to $x = 6$, then V_y is a linear function in x (a straight line) from $x = 0$ to $x = 6$.

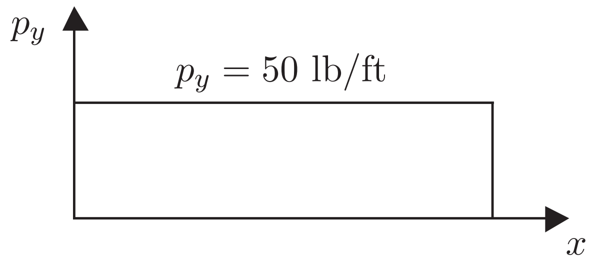


Figure 13.27:

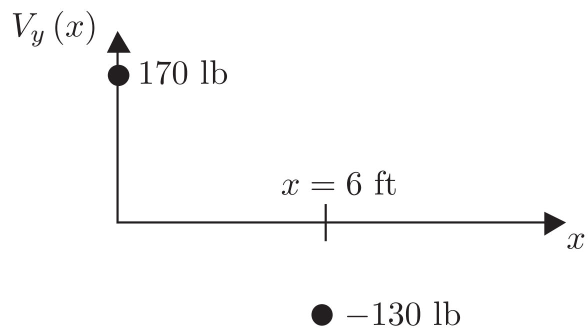


Figure 13.28:

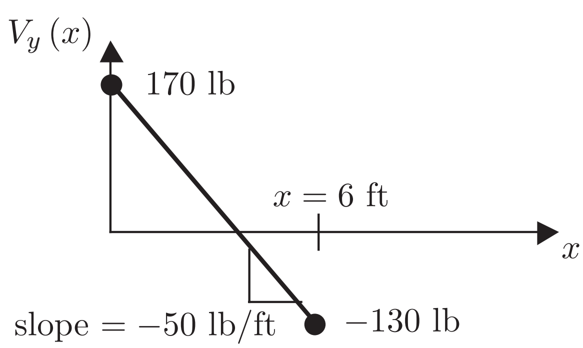


Figure 13.29:

Also the slope of the shear curve is equal to $-p_y$ or $-50\ \frac{\text{lb}}{\text{ft}}$ from $x = 0$ to $x = 6$. The equation of this curve is easily obtained: $V_y(x) = 170 - 50x$ lb. Note that the shear is zero at $x = 3.4$ ft.

For the moment, we write

$$\begin{aligned} M_z(6) &= M_z(0) - \text{area under the } V_y \text{ curve from 0 to 6} \\ &= 0 - \left[\frac{1}{2}(3.4\text{ ft})(170\text{ lb}) + \frac{1}{2}(2.6\text{ ft})(-130\text{ lb})\right] = -120\text{ ft - lb} \end{aligned}$$

This gives us two points on the M_z curve to plot $M_z(0) = 0$ and $M_z(6) = -120$ ft-lb:

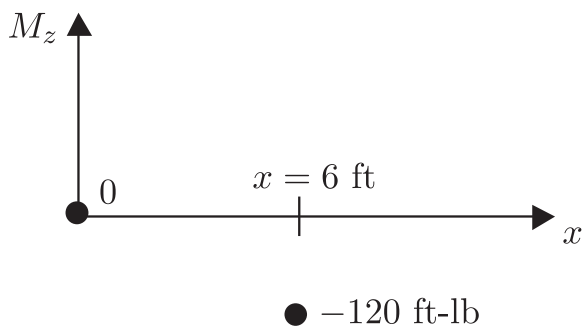


Figure 13.30:

Since the shear is linear from $x = 0$ to $x = 6$, then the moment is quadratic (a parabola). To determine whether the parabola is convex upward or downward, consider equation (13.21) $\frac{d^2M_z}{dx^2} = -\frac{dV_z}{dx} = p_y$. For a positive p_y, $\frac{d^2M_z}{dx^2} > 0$, which means that the M_z curve is convex upwards. In our case, $p_y = +50\ \frac{\text{lb}}{\text{ft}}$ and therefore the curve is convex upwards. From $\frac{dM_z}{dx} = -V_y$, we know the moment is a max or min where $V_y = 0$. Setting the shear equal to zero and solving for x gives $x = 3.4$ ft. $M_z(3.4) = 0 - \frac{1}{2}(3.4)(170)$ ft - lb. We now sketch the curve:

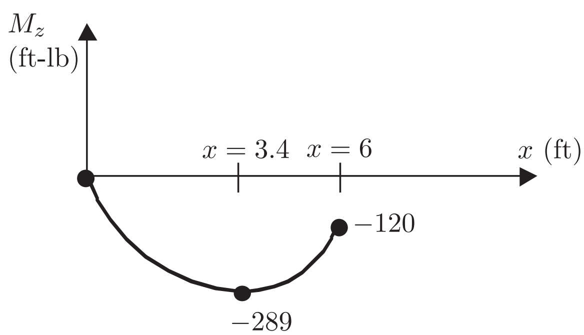


Figure 13.31:

The remaining portion of the shear and moment (from $x = 6$ to $x = 10$) may be obtain in a similar manner to obtain the complete shear and moment diagrams.

Example 13-5

Consider the cantilever beam in Example problem 13-4 and assume the beam has the cross-section shown in Example 13-2.

a) Determine the bending stress at $x = 6$ ft (top surface) and (bottom surface).

b) Determine the location and magnitude of the maximum bending stress. Indicate whether it is tensile or compressive.

From example 13-2, the cross-section and location of the centroid are shown in the sketch below. The moment of inertia was determined to be $I_{zz} = 2.31 \times 10^6$ mm^4.

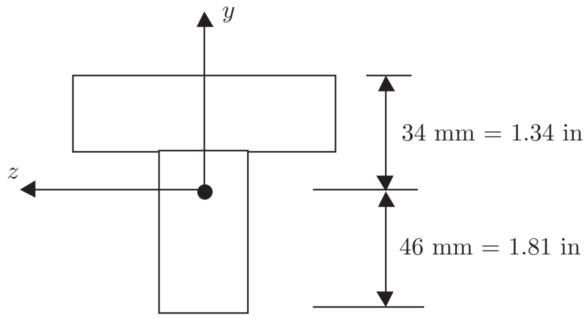


Figure 13.32:

Converting the moment of inertia to English units gives:

$$I_{zz} = 3.31 \times 10^6 \text{ mm}^4 \left(\frac{\text{in}}{25.4 \text{ mm}}\right)^4 = 7.95 \text{ in}^4$$

a) From the moment diagram in Example 13-4, $M_z(6 \text{ ft}) = 120$ ft-lb $= -1,440$ in-lb. The bending stress is given by equation (13.13)

$$\sigma_{xx} = -\frac{M_z y}{I_{zz}}$$

At the top surface, $y = 1.34$ in, and the bending stress is equal to

$$\sigma_{xx}(\,72'', 1.34''\,) = -\frac{M_z y}{I_{zz}} = -\frac{(-1,440 \text{ in - lb})(1.34 \text{ in})}{7.95 \text{ in}^4} = 243 \text{ psi (tension)}$$

At the bottom surface, $y = -1.81$ in, and the bending stress is equal to

$$\sigma_{xx}(\,72'', -1.81''\,) = -\frac{M_z y}{I_{zz}} = -\frac{(-1,440 \text{ in- lb})(-1.81 \text{ in})}{7.95 \text{ in}^4} = -328 \text{ psi (compression)}$$

b) For a prismatic beam, the maximum bending stress will occur at the location of maximum bending moment. From Example 13-4, the maximum bending moment occurs at $x = 3.4$ ft $= 40.8$ in, and is equal to $M_z = -289$ ft-lb $= -3,468$ in-lb. The bending stress at this point is given by

$$\sigma_{xx}(\,40.8'', y\,) = -\frac{M_z y}{I_{zz}} = -\frac{(\,-3,468 \text{ in - lb}\,)y}{7.95 \text{ in}^4} = 436.2y \text{ psi}$$

The last result shows that at $x = 3.4$ ft (40.8 in), the bending stress is tensile for positive y (above the centroidal axis) and compressive for negative y. Thus the *maximum tensile stress* will occur at the upper surface where $y = 1.34$ in and is given by

$$\sigma_{xx}(40.8'', 1.34'') = 436.2y \text{ psi} = 436.2(1.34) \text{ psi} = 585 \text{ psi (maximum tensile stress)}$$

Note that in determining the bending stress, we have not used any material properties (Young's modulus, E, for beam bending). For statically determinate structures like the cantilever beam considered here, the stress is always independent of material properties since the internal stress solution may be obtained from equilibrium equations alone. However, the majority of structures are *statically indeterminate*, and the stress solution will depend on material properties. Several examples of statically indeterminate uniaxial bar problems were presented in Chapter 11 (see Examples 11-2 and 11-3).

Example 13-6

Given the following cantilevered bar with a distributed and applied shear load, determine through integration methods the equations for the shear (V_y), moment (M_z), and displacement (u_{0y}). Also, draw the shear and moment diagrams.

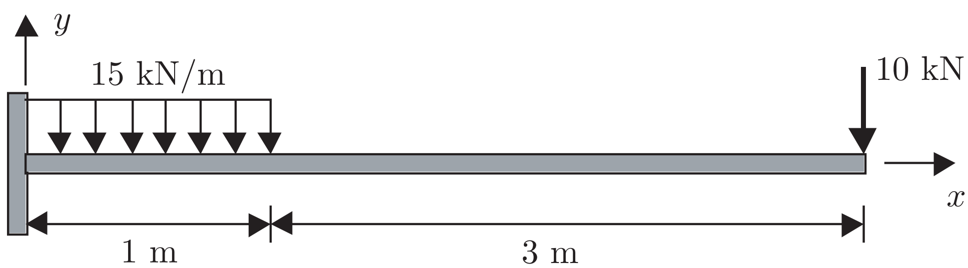


Figure 13.33:

We first determine the reactions as the wall. Draw a free-body diagram labeling the shear and moment reaction at the wall:

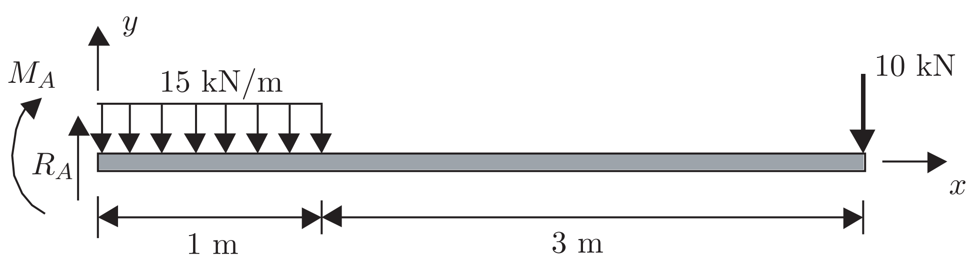


Figure 13.34:

Use conservation of linear and angular momentum to obtain the reactions:

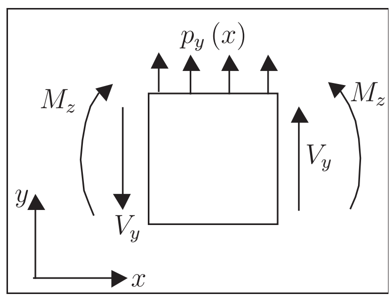
$$\begin{aligned}
0 &= \sum F_x = R_A - \left(15\ \frac{\text{kN}}{\text{m}}\right)(1\ \text{m}) - 10\ \text{kN} \\
\therefore R_A &= 25\ \text{kN} \\
0 &= \sum M_z\ (\text{about left end}) = \left(15\ \frac{\text{kN}}{\text{m}}\right)(1\ \text{m})(0.5\ \text{m}) + (10\ \text{kN})(4\ \text{m}) + M_A \\
\therefore M_A &= -47.5\ \text{kN - m}
\end{aligned}$$

Now use equation (13.18) [or (13.22)] to obtain the shear distribution.
For $0 < x < 1$ m:

$$\frac{dV_y}{dx} = -p_y = -\left(-15\ \frac{\text{kN}}{\text{m}}\right) = 15\ \frac{\text{kN}}{\text{m}} \quad (\text{due to the distributed load - note sign of } p_y)$$

By integration: $V_y = 15x + C$
at $x = 0$, $V_y = -R_A = -25$ kN, $\therefore C = -2$
$\therefore V_y(x) = 15x - 25$ kN

In evaluating the shear and moment, and the constants of integration, **it is imperative to keep in mind the sign convention for shear and bending moment that was established earlier**:



Sign Convention for V and M

Figure 13.35:

For $1 < x < 4$ m:

$$\frac{dV_y}{dx} = 0 \quad (\text{distributed load is zero})$$

By integration: $V_y = C$
at $x = 1$, from the shear equation for $0 \le x \le 1$, $V_y(1) = 15(1) - 25 = -10$ kN

$\therefore V_y = -10$ kN $= C$ $\therefore V_y(x) = -10$ kN
Now we can determine the moment distribution using equation (13.19):
For $0 < x < 1$ m:

$$\frac{dM_z}{dx} = -V_y(x) = -15x + 25\ \text{kN}$$

By integration: $M_z(x) = -7.5x^2 + 25x + c$

at $x = 0$, $M_z(0) = M_A = -47.5$ kN - m
$\therefore M(x) = -7.5x^2 + 25x - 47.5$ (kN - m)
For $1 < x < 4$ m:

$$\frac{dM_z}{dx} = -V_y(x) = 10 \text{ kN}$$

By integration: $M_z(x) = 10x + C$
at $x = 1$, from the moment equation for $0 \le x \le 1$, $M_z(1) = -30$ kN - m
$\therefore M_z(1) = -30 = 10(1) + C \therefore C = -40$
$\therefore M_z(x) = 10x - 40$ (kN - m)
We can now draw the shear and moment distribution for the beam using the equations above.
The last step is to determine the displacement equation.
For $0 < x < 1$ m:

$$\begin{aligned} \frac{d^2u_{0y}}{dx^2} &= \frac{M_z(x)}{EI_{zz}} = \frac{-7.5x^2 + 25x - 47.5}{EI_{zz}} \implies \frac{du_{0y}}{dx} = \frac{1}{EI_{zz}}\left(-\frac{7.5x^3}{3} + 25\frac{x^2}{2} - 47.5x + c_3\right) \\ u_{0y} &= \frac{1}{EI_{zz}}\left(-\frac{7.5x^4}{12} + 25\frac{x^3}{6} - \frac{47.5}{2}x^2 + c_3x + c_4\right) \end{aligned}$$

at $x = 0$, the beam is fixed from displacement and rotation (cantilever beam)

$$\begin{aligned} u_{0y}(x=0) = 0 &\implies c_4 = 0, \\ \frac{du_{0y}}{dx}(x=0) = 0 &\implies c_3 = 0 \end{aligned}$$

$$u_{0y} = \frac{1}{EI_{zz}}\left(-\frac{7.5x^4}{12} + 25\frac{x^3}{6} - \frac{47.5}{2}x^2\right) \text{ m} \qquad 0 < x < 1 \text{ m}$$

For $1 < x < 4$ m:

$$\begin{aligned} \frac{d^2u_{0y}}{dx^2} &= \frac{M_z(x)}{EI_{zz}} = \frac{10x - 40}{EI_{zz}} \implies \frac{du_{0y}}{dx} = \frac{1}{EI_{zz}}\left(\frac{10x^2}{2} - 40x + c_1\right) \\ u_{0y} &= \frac{1}{EI_{zz}}\left(\frac{10x^3}{6} - \frac{40x^2}{2} + c_1x + c_2\right) \end{aligned}$$

Apply the boundary condition at $x = 1$. Evaluate the the displacement at $x = 1$ using the solution for $0 \le x \le 1$ and set equal to the solution for $1 \le x \le 4$ at $x = 1$:

$$\begin{aligned} u_{0y}|_{x=1} &= \frac{1}{EI_{zz}}\left(-\frac{7.5(1)^4}{12} + \frac{25(1)^3}{6} - \frac{47.5(1)^2}{2}\right) = \frac{-20.2}{EI_{zz}} = \frac{1}{EI_{zz}}\left(\frac{10(1)^3}{6} - \frac{40(1)^2}{2} + c_1(1) + c_2\right) \\ \left.\frac{du_{0y}}{dx}\right|_{x=1} &= \frac{1}{EI_{zz}}\left(-\frac{7.5(1)^3}{3} + \frac{25(1)^2}{2} - 47.5(1)\right) = \frac{-37.5}{EI_{zz}} = \frac{1}{EI_{zz}}\left(5(1)^2 - 40(1) + c_1\right) \end{aligned}$$

Thus we have two equations defining the constants of integration:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} -1.867 \\ -2.5 \end{Bmatrix}$$

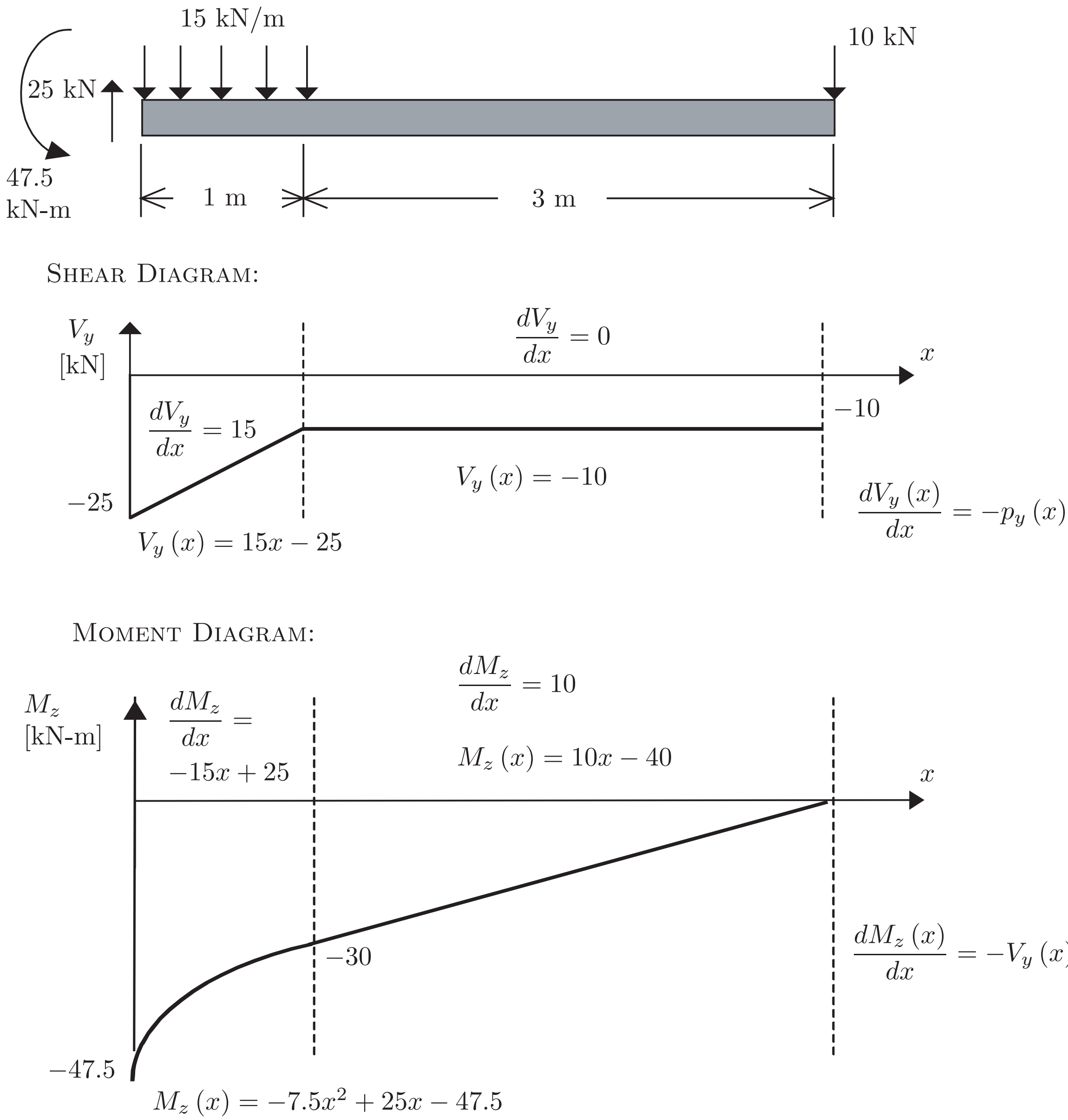


Figure 13.36:

and the solution is:

$$
\begin{aligned}
c_1 &= -2.5 \\
c_2 &= 0.633
\end{aligned}
$$

Hence the displacement is given by

$$u_{0y} = \frac{1}{EI_{zz}}\left(\frac{5x^3}{3} - 20x^2 - 2.5x + 0.633\right) \text{ m} \qquad 1 < x < 4 \text{ m}$$

Note that I_{zz} must be in m^4 and E in $\frac{\text{N}}{\text{m}^2}$ or MPa.

Example 13-7

Plot $u_{0y}(EI_z)$ for *Example 13-6* using Scientific Workplace.

Solution

$$g(x) = \begin{cases} -\frac{7.5x^4}{12} + 25\frac{x^3}{6} - \frac{47.5}{2}x^2 & \text{if} \quad 0 < x \leq 1 \\ \frac{5x^3}{3} - 20x^2 - 2.5x + 0.633 & \text{if} \quad 1 \leq x < 4 \end{cases}$$

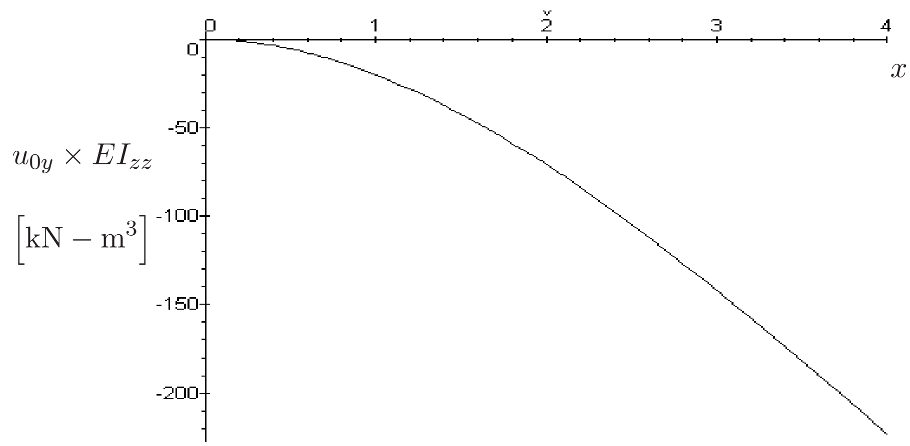


Figure 13.37:

Important note: Because we are plotting the displacement scaled by EI_{zz} (which is typically a large value), the above displacement "looks" very large. The actual displacement $u_{0y}(x)$ would in fact be typically quite small (and usually difficult to see with the eye).

Example 13-8

If the cantilevered bar from *Example 13-6* is assumed to be aluminum with a rectangular cross-section, find the moment of inertia and the amount of deflection for the two cases:

1) base = 0.01 m, height = 0.1 m

2) base = 0.05 m, height = 0.2 m

Solution

Case 1):

$$\left.\begin{array}{l} I_{zz} = \frac{1}{12}(0.01)(0.1)^3 = \frac{1}{12}\times 10^{-5}\ \mathrm{m}^4 \\ E_{\mathrm{Al}} = 70\ \mathrm{GPa} = 70\times 10^9\ \frac{\mathrm{N}}{\mathrm{m}^2} \end{array}\right\} \begin{array}{l} EI_{zz} = \frac{70}{12}10^4\ \mathrm{N}\cdot\mathrm{m}^2 \\ \quad = \frac{700}{12}\ \mathrm{kN}\cdot\mathrm{m}^2 \end{array}$$

$$u_{0y}\big|_{x=4} = \frac{-222.7}{EI_z} = \frac{-222.7}{\frac{700}{12}} = 3.8\ \mathrm{m} \qquad \textit{very large deflection!}$$

Case 2):

$$\begin{aligned} I_{zz} &= \frac{1}{12}(0.05)(0.2)^3 = \frac{1}{3}\times 10^{-4}\ \mathrm{m}^4 \\ EI_{zz} &= \frac{70}{3}10^5\ \mathrm{N\ m}^2 = \frac{7000}{3}\ \mathrm{kN}\cdot\mathrm{m}^2 \\ u_{0y}\big|_{x=4} &= \frac{-222.7}{EI_z} = \frac{-222.7}{\frac{7000}{3}} = 0.095\ \mathrm{m} \end{aligned}$$

Example 13-9

Given: The following bar is cantilevered at its right end and carries a distributed and applied load as shown below. Determine by integration methods the equations for the shear (V_y) and moment (M_z). Also, draw the Shear/Moment diagrams for the bar.

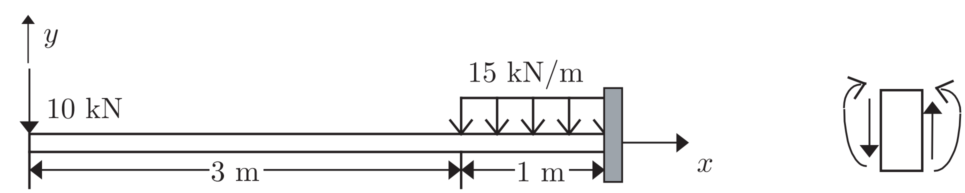


Figure 13.38:

Solution

Draw the free-body diagram with shear and moment reaction at the right end.
Apply conservation of linear and angular momentum to determine the reactions:

Sum the forces.

$$\sum F = 0 = RA - \left(15\ \frac{\mathrm{kN}}{\mathrm{m}}\right)(1\ \mathrm{m}) - 10\ \mathrm{kN}$$

$RA = 25\ \mathrm{kN}$

Sum the moments.

$$\sum M = 0 = \left(15\ \frac{\mathrm{kN}}{\mathrm{m}}\right)(1\ \mathrm{m})(0.5\ \mathrm{m}) + (10\ \mathrm{kN})(4\ \mathrm{m}) + MA$$

$MA = -47.5\ \mathrm{kN\ m}$

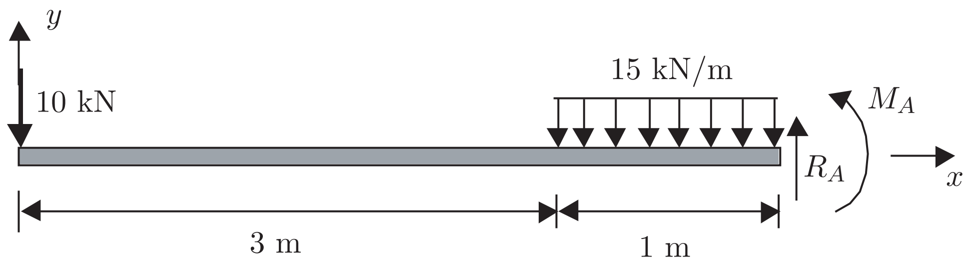


Figure 13.39:

Next, use integration to determine the shear/moment for each section.
$0 \le x \le 3$ m

$$\frac{dV_y}{dx} = -p_y = 0 \text{ (due to no distributed load)}$$

By integration: $V_y = C$
at $x = 0$, $V_y = 10 \rightarrow C = 10 \quad \therefore V_y(x) = 10$ kN

$$\frac{dM_z}{dx} = -V_y(x) = -10 \text{ kN}$$

By integrtion: $M_z(x) = -10x + C$

$$M_z(0) = 0 \rightarrow C = 0 \quad \therefore M_z(x) = -10x \text{ (kN - m)}$$

$3 < x < 4$ m

$$\frac{dV_y}{dx} = -p_y = -\left(-15\ \frac{\text{kN}}{\text{m}}\right) = 15\ \frac{\text{kN}}{\text{m}} \text{ (due to the distributed load)}$$

By integration: $V_y = 15x + C$
at $x = 3$ m, $V_y = 10$ kN (from $0 < x < 3$ solution for shear) $\rightarrow C = -35 \therefore V_y(x) = 15x - 35$

$$\frac{dM_z}{dx} = -V_y(x) = -15x + 35 \text{ kN}$$

By integration: $M_z(x) = -7.5x^2 + 35x + C$
$M_z(3) = -30\ \frac{\text{kN}}{\text{m}}$ (from $0 < x < 3$ solution for moment) $\rightarrow C = -67.5$
$\therefore M_z(x) = -7.5x^2 + 35x - 67.5\ \left(\frac{\text{kN}}{\text{m}}\right)$
The shear and moment diagrams may now be drawn using the shear and bending moment equations derived above.

13.5 Shear Stress in Beam Bending

For the case of pure bending, we considered only the axial stress σ_{xx} under the assumption that all shear stresses were zero. The conservation of linear momentum equation reduced to $\frac{\partial \sigma_{xx}}{\partial x} = 0$

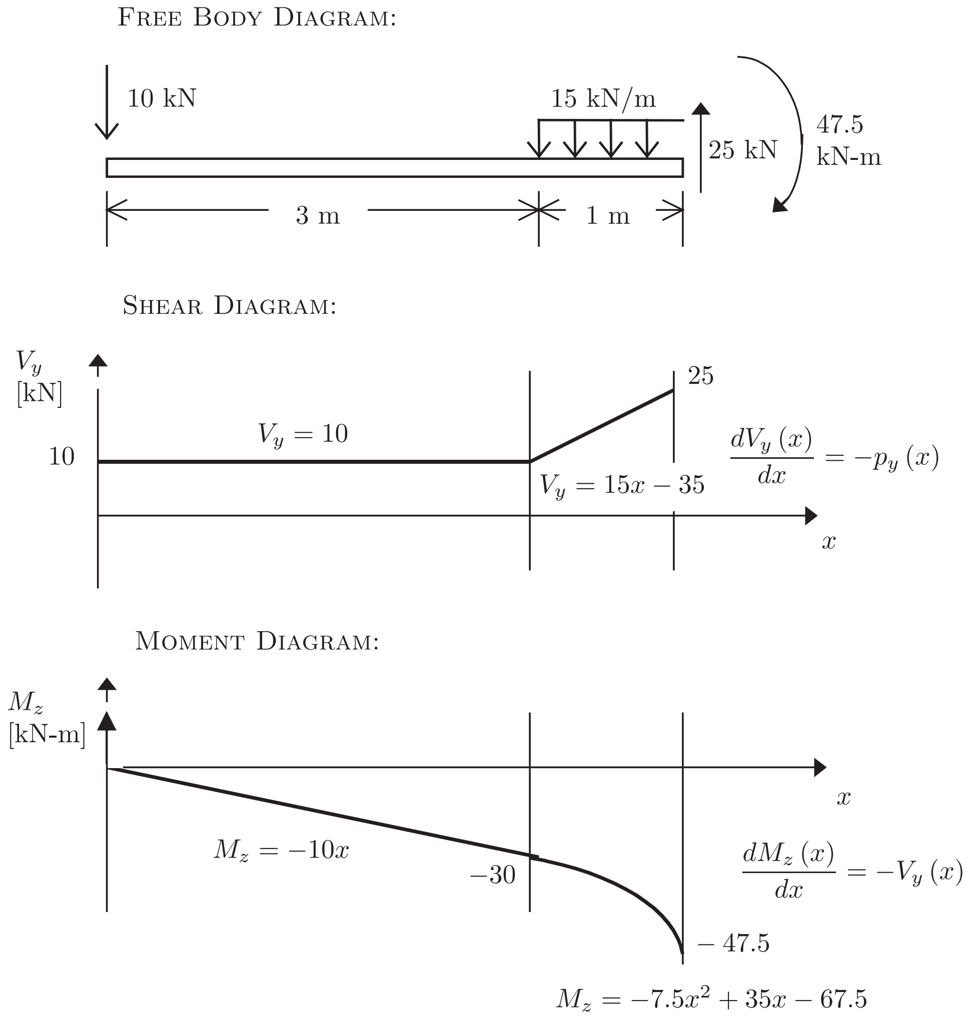


Figure 13.40:

which gives a solution of σ_{xx} = constant. However, as we have seen in previous examples dealing with shear and moment diagrams, when transverse loads p_y or transverse shear forces are applied to the structure, the moment distribution is no longer a constant. Hence the axial stress σ_{xx} is now a function of x. Because the axial stress is not constant, a shear stress σ_{xy} must exist on the cross-section in order to satisfy global equilibrium. We wish to determine this shear stress distribution that exists at a cross section located at any point x. We first construct a free-body of the beam at any point x by passing a horizontal cutting a plane at a distance $y = h$ and two vertical cutting planes at point x and $x + \Delta x$, respectively, as shown below.

Note that the figure is drawn with the y-axis pointing downwards for visualization purposes only and that no sign conventions have been changed. Although an almost rectangular cross section is shown, note that the cross section may be any shape. On the top surface we see a shear stress which will be a function of position y. On each end face we see an axial stress σ_{xx} as well as a shear stress σ_{xy}. We can apply conservation of linear momentum in the x direction by summing forces due to the stresses in the x direction for the cross section located from any point $y = h$ to the outer most

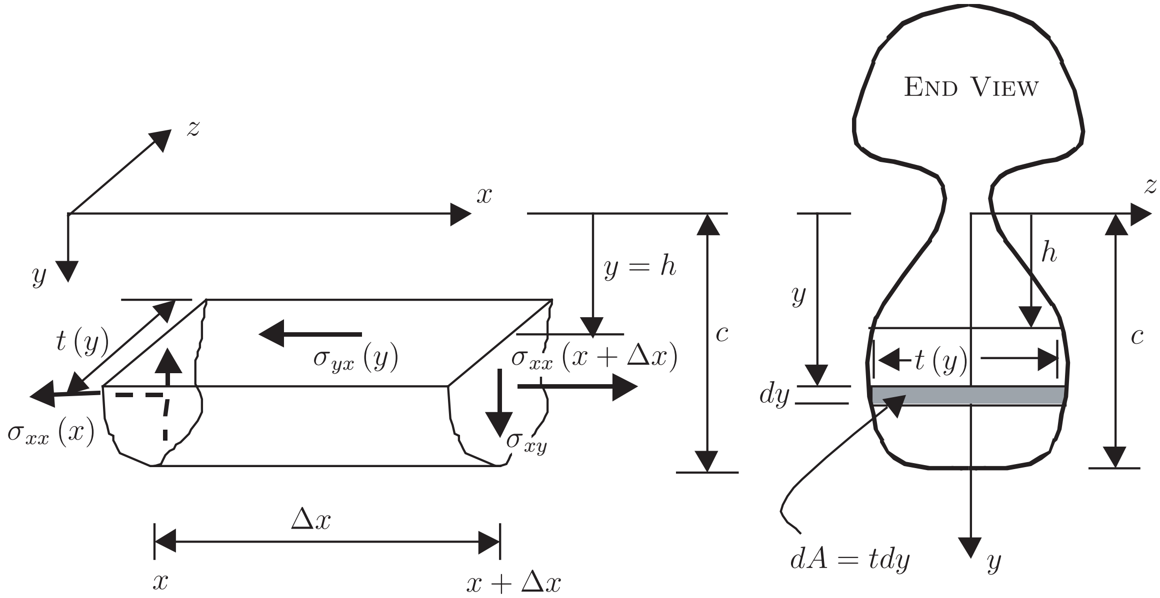


Figure 13.41: Free-Body of Beam at $(x, y = h)$

point $(y = c)$ of the beam. This gives the following *equilibrium equation*:

$$0 = \sum F_x = -\int_x^{x+\Delta x} \sigma_{yx}(x, y = h)t(y)\,dx + \int_h^c \sigma_{xx}(x + \Delta x, y)t(y)\,dy - \int_h^c \sigma_{xx}(x, y)t(y)\,dy \tag{13.25}$$

Note that the bending stress and width of the beam (t) are functions of y. Divide the above equation by Δx to obtain:

$$\sigma_{yx}(x, h)t(y) = \int_h^c \frac{\sigma_{xx}(x + \Delta x, y) - \sigma_{xx}(x, y)}{\Delta x} t(y)\,dy \tag{13.26}$$

Take the limit as $\Delta x \to 0$ for the stress term to obtain

$$\lim_{\Delta x \to 0} \frac{\sigma_{xx}(x + \Delta x, y) - \sigma_{xx}(x, y)}{\Delta x} = \frac{d\sigma_{xx}(x, y)}{dx} \tag{13.27}$$

Thus equation (13.26) becomes

$$\sigma_{yx}(x, h)t(y) = \int_h^c \frac{d\sigma_{xx}}{dx} t(y)\,dy \tag{13.28}$$

The stress term in the last equation can be written in terms of the bending moment by using equation (13.13) to obtain

$$\frac{d\sigma_{xx}}{dx} = \frac{d\left(-\frac{M_z y}{I_{zz}}\right)}{dx} = -\frac{y}{I_{zz}} \frac{dM_z}{dx} \tag{13.29}$$

Now introduce the relation between bending moment and shear, equation (13.20), into the above to obtain

$$\frac{d\sigma_{xx}}{dx} = -\frac{y}{I_{zz}}(-V_y) \tag{13.30}$$

With the above result, equation (13.28) becomes

$$\sigma_{yx}(\,x,h\,)t(y) = \int_h^c \frac{y}{I_{zz}} V_y t(y)\,dy \tag{13.31}$$

The shear and moment of inertia terms may be taken outside the integral since they are functions of x only. Hence, the last result may be written

$$\sigma_{yx}(\,x,h\,)t(y) = \frac{V_y(x)}{I_{zz}} \int_h^c t(y)y\,dy \tag{13.32}$$

Dividing by the width $t(y)$ gives

$$\sigma_{yx}(\,x,h\,) = \frac{V_y(x)}{I_{zz}t(y)} \int_h^c t(y)y\,dy \tag{13.33}$$

The integral term is a geometrical property so that the last result may be written

$$\sigma_{yx}(\,x,h\,) = \frac{V_y(x)}{I_{zz}t(y)} Q(h) \tag{13.34}$$

where Q is called the first moment of the area and given by

$$Q(h) \equiv \int_h^c yt(y)\,dy \tag{13.35}$$

Equation (13.34) provides the magnitude of the shear stress at any distance $y = h$ from the centroidal axis. Note that at $y = \pm c$ (top or bottom surface of the beam), $Q = 0$ and hence the shear stress is zero at the top and bottom locations of the cross-section. For a rectangular cross section, we will show in a later example that the shear stress varies quadraticly over the cross section and is a maximum at the centroid of the cross-section ($y = 0$).

The integral equation defining $Q(h)$ can be simplified for simple composite cross-sections consisting of rectangles or other simple shapes for which the centroid is known. Consider the cross-section shown below:

We want to determine $Q(h) \equiv \int_h^c yt(y)\,dy$ for the shaded area in Figure 13.18(a). Note from Figure 13.18(b) that this integral can be written as $Q(h) \equiv \int_h^c yt(y)\,dy = \int_h^c y\,dA$ where $dA = tdy$. For the differential area dA in Figure 13.18(b), y can be taken as the distance to the centroid of dA, i.e., $y = \bar{y}$ for dA. Hence, for finite size areas, we can write the integral as a summation:

$$Q(h) \equiv \int_h^c yt(y)\,dy = \int_h^c y\,dA = \sum_h^c \bar{y}_i A_i \tag{13.36}$$

As an example, consider the T section below. We wish to determine $Q(0)$, i.e., at the centroid. We divide the area above $y = 0$ into two areas as shown. From Equation (13.36), we obtain:

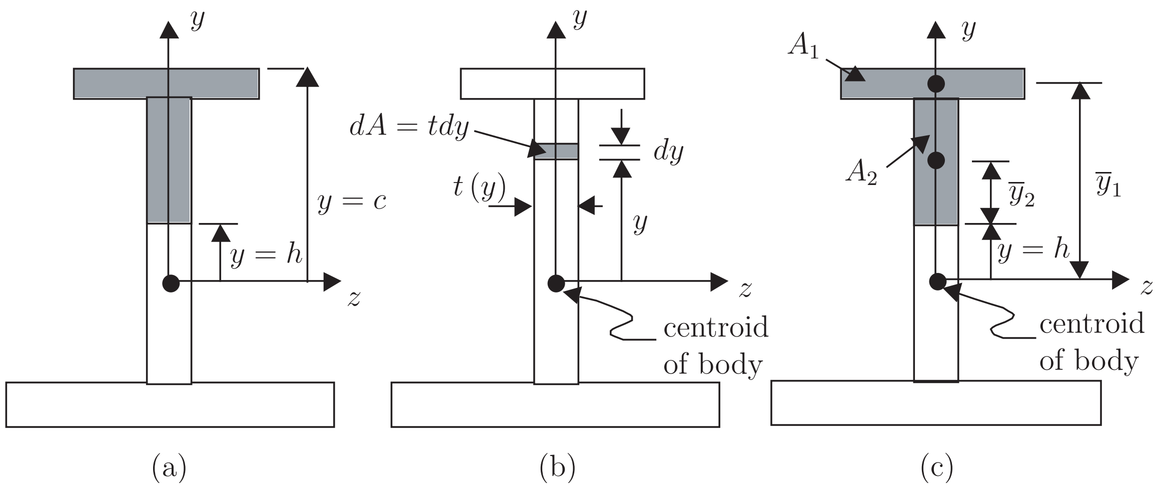


Figure 13.42: Geometry for Determination of Q (first moment of area)

$$
\begin{aligned}
Q(0) &= \bar{y}_1 A_1 + \bar{y}_2 A_2 \\
&= 24 \text{ mm}(80 \text{ mm} \times 20 \text{ mm}) + 7 \text{ mm}(40 \text{ mm} \times 14 \text{ mm}) \\
&= 42,300 \text{ mm}^3
\end{aligned}
$$

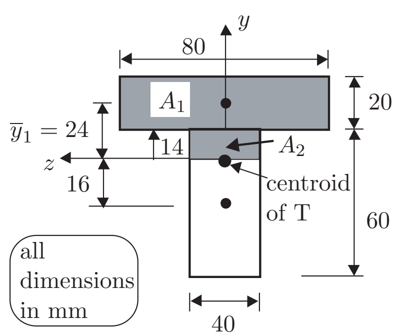


Figure 13.43:

Alternately, we could have taken the area below $y = 0$ to obtain the same result:

$$
\begin{aligned}
Q(0) &= \bar{y}_1 A_1 \\
&= -23 \text{ mm}(-40 \text{ mm} \times 46 \text{ mm}) = 42,300 \text{ mm}^3
\end{aligned}
$$

Note that the area is negative in the equation above because the area is below the z axis. Examples of both methods for determining Q may be found in the examples below.

Example 13-10

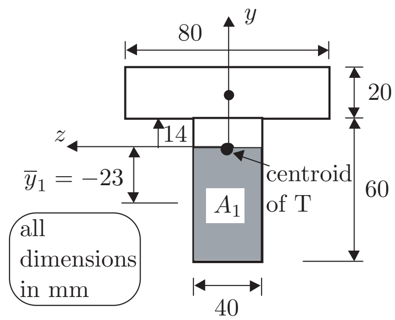


Figure 13.44:

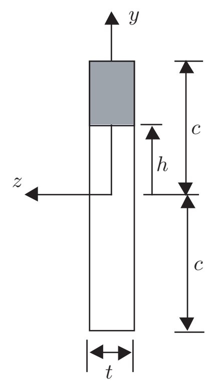


Figure 13.45:

Consider a rectangular cross-section of width t and height $2c$ Now calculate $Q(h)$:

$$Q(y = h) = \int_h^c yt\,dy = t\int_h^c y\,dy = \frac{t}{2}(c^2 - h^2)$$

Note: at top or bottom, $Q(y = \pm c) = 0$
at center, $Q(y = 0) = \frac{t}{2}c^2 = \frac{c}{2}(tc)$

Thus, Q is 0 at the top and bottom, is a maximum at centroid, and varies quadraticly from top to bottom.

For the rectangular cross section, σ_{xy} is a maximum at centroid and is given by:

$$\sigma_{xy}(\,x, y = 0\,) = \frac{3}{4}\frac{V_y(x)}{tc}.$$

Example 13-11

Consider a beam with the cross-section shown below. The centroid is located 36 mm from the top surface (or 46 mm from the bottom surface) and the moment of inertia about the z axis is determined to be $I_z = 2.31 \times 10^6$ mm^4 (see Example 13-2). Assume that at some point along the length of the beam, the shear has a value of $V_z = 10 \times 10^3$ N. Determine the shear stress at $y = +14$ mm and $y = -14$ mm.

Given:

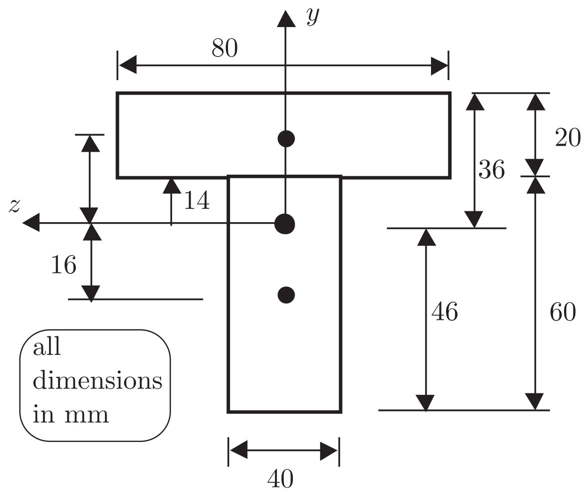


Figure 13.46:

Solution

The shear stress at any point is given by: $\sigma_{xy}(x, y) = \frac{V_y(x)Q(y)}{I_{zz}b(y)}$

Substituting the values of V_y and I_{zz} gives: $\sigma_{xy} = \frac{10\times10^3[\text{N}]Q(y)}{2.31\times10^6[\text{mm}^4]b(y)}$

At $y = 14$ mm: $Q(14 \text{ mm}) = \int_{14}^{34} yt(y)\,dy = \int_{14}^{34} y(80)\,dy = \frac{y^2}{2}80\Big|_{14}^{34} = 38.4 \times 10^3$ [mm^3]

Alternately, we could determine $Q(14$ mm$)$ using Equation (13.36). Divide the area above $y = 14$ mm into two areas as shown to the right.

Substituting $Q(14$ mm$)$ in the shear stress equation gives:

$$\sigma_{xy}(y = 14 \text{ mm}) = \frac{10 \times 10^3[\text{N}]\ 38.4 \times 10^3[\text{mm}^3]}{2.31 \times 10^6[\text{mm}^4]\ 80[\text{mm}]} = 2.079\ \frac{\text{N}}{\text{mm}^2} = 2.079 \times 10^6\ \frac{\text{N}}{\text{m}^2} = 2.079 \text{ MPa}$$

Similarly, $\sigma_{xy}(y = -14 \text{ mm}) = \frac{10\times10^3[\text{N}]\ 38.4\times10^3[\text{mm}^3]}{2.31\times10^6[\text{mm}^4]\ 40[\text{mm}]} = 4.157\ \frac{\text{N}}{\text{mm}^2} = 4.157$ MPa

The shear stress is of course a maximum at the centroid. $Q(0)$ is given by:

$$Q(0) = \bar{y}_1 A_1 = -23(-40 \times 46) = 42,300 \text{ mm}^3$$

Note that the area is negative in the equation above because the area is below the z axis. The shear stress becomes:

$$\sigma_{xy}(y = 0) = \frac{10 \times 10^3[\text{N}]\ 42.3 \times 10^3[\text{mm}^3]}{2.31 \times 10^6[\text{mm}^4]\ 40[\text{mm}]} = 4.578\frac{\text{N}}{\text{mm}^2} = 4.578 \text{ MPa}$$

At the upper surface, $Q(34 \text{ mm}) = 0$ and at the lower surface, $Q(-46 \text{ mm}) = 0$ which means that the shear stress is zero at the upper and lower surface. For example,

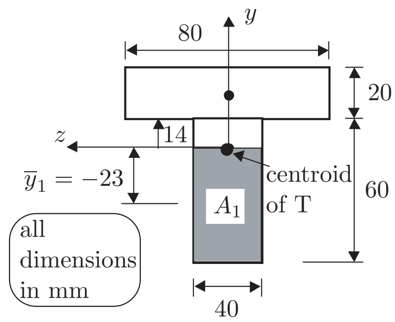


Figure 13.47:

$$Q(-46\text{ mm}) = Q(14\text{ mm}) + \int_{-46}^{14} y40\,dy = 38.4 \times 10^3 + 40\frac{y^2}{2}\bigg|_{-46}^{14} = 0$$

Consider another cross-section. Assume the same shear force and evaluate the shear stress at the centroid of the rectangular cross-section.

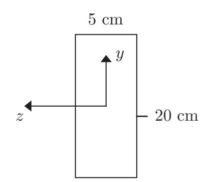
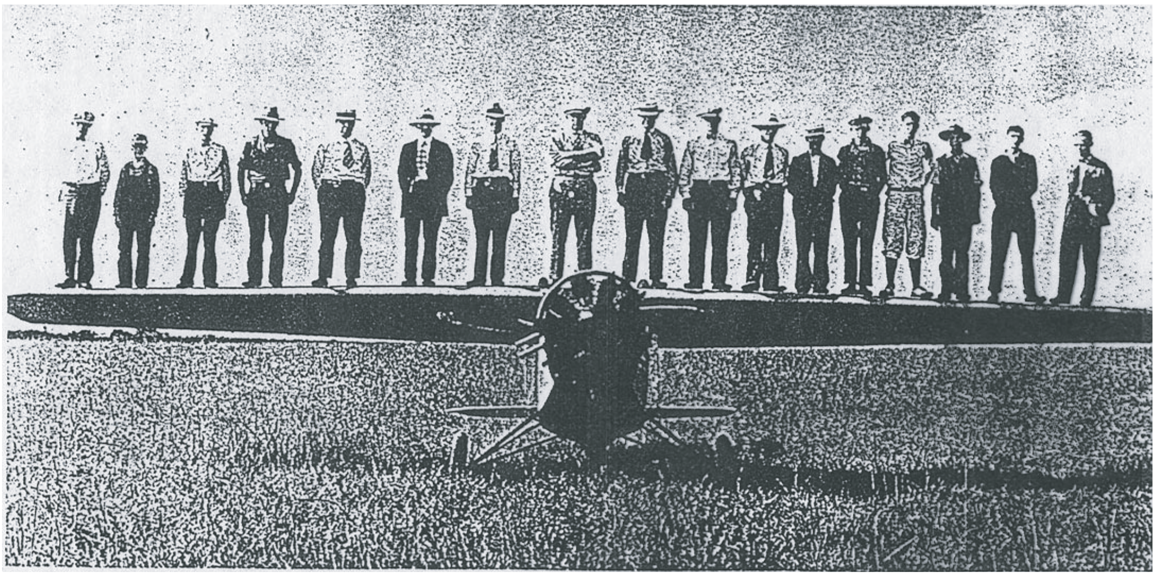


Figure 13.48:

Deep Thought



17 men stand atop a prototype strutless wing design
to demonstrate its strength.

13.6 Questions

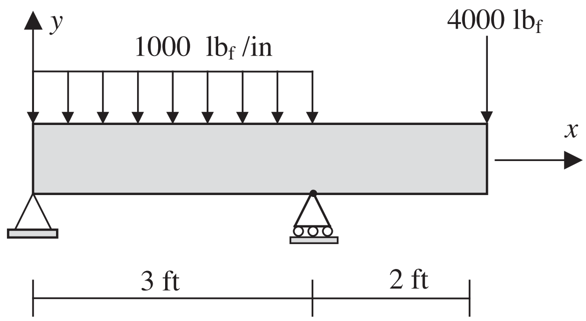
13.1 How is a shear stress created from a shear force and how are tensile and compressive forces created from bending moments? (Hint: Examine the properties of a thin, rubber eraser)

13.2 How are the properties of the cross section of a beam incorporated into the equations representing the shear and normal stresses for a beam?

13.7 Problems

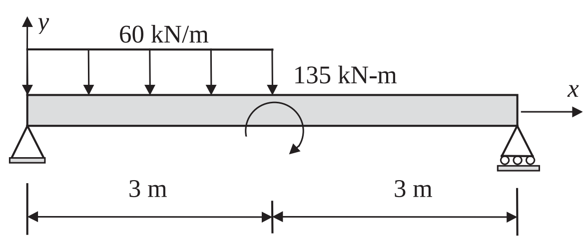
13.3 Determine the equations for the shear, $V(x)$, and moment, $M(x)$, in each of the following beams. Plot the functions along the x-axis. Use the given coordinate system for your answer.

a.



Problem 13.3 a

b.



Problem 13.3 b

c.

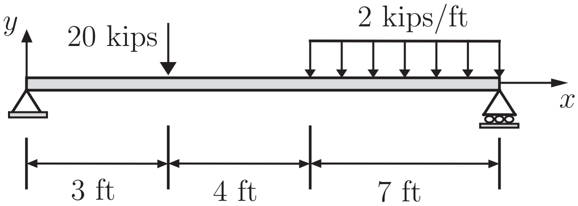
d.

e.

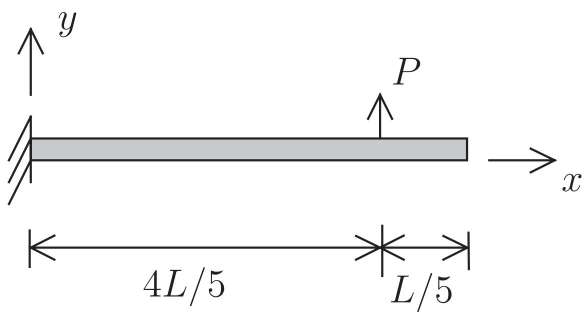
f.

g.

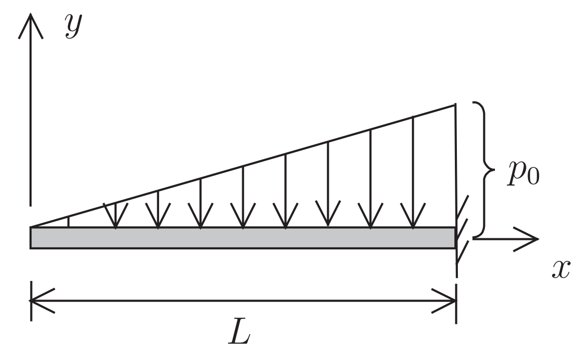
h.



Problem 13.3 c



Problem 13.3 d



Problem 13.3 e

i.

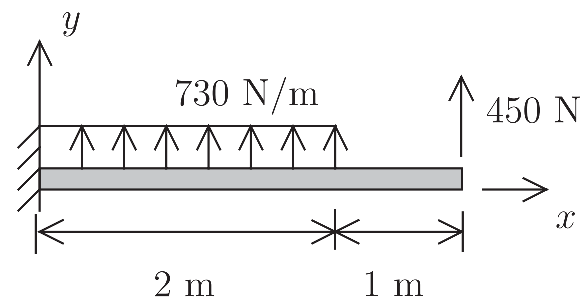
j.

k.

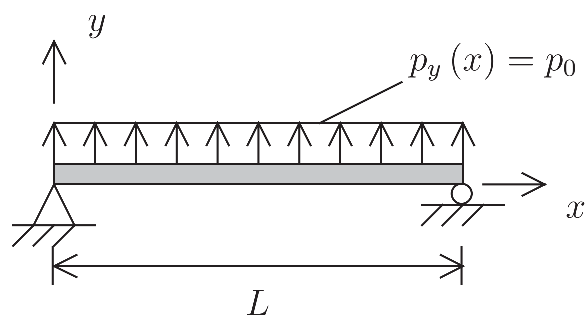
l.

m.

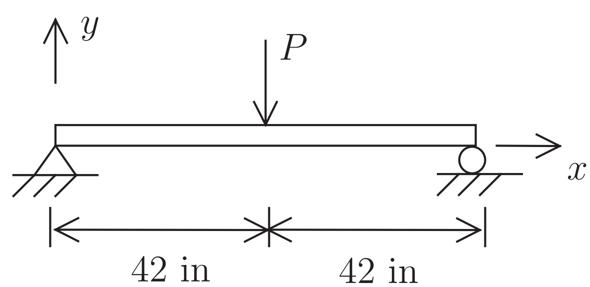
13.4 Determine the stress distributions along the top of the beam for parts a, c, and f in problem 13.3. Assume the following rectangular cross-section.



Problem 13.3 f



Problem 13.3 g



Problem 13.3 h

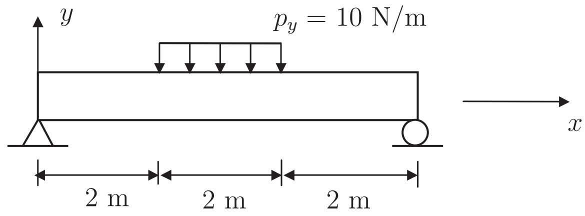
13.5 *GIVEN*: $M = 5000\ \mathrm{lb_f \cdot ft}$

DETERMINE: The moment of Inertia of the rectangle ($I = \frac{1}{12}ba^3$)

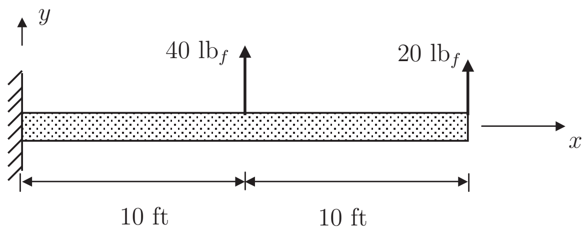
13.6 *GIVEN*: The two cantilevered beams loaded as shown in the figure below,

SELECT: The case that is better designed to support the applied load. Both beams have identical cross-sections and material makeup. Justify your selection.

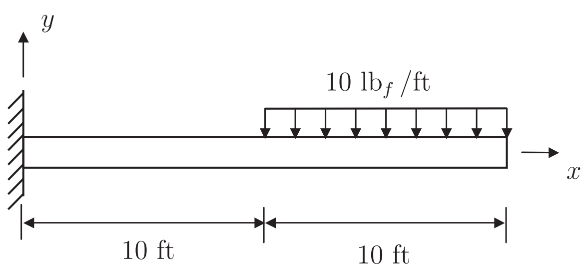
13.7 *GIVEN*: The beam shown.



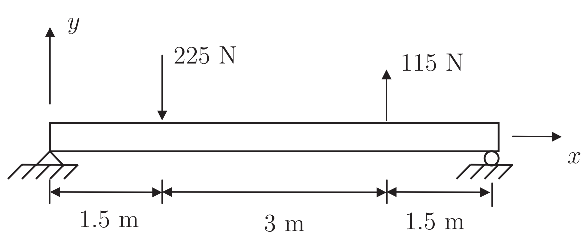
Problem 13.3 i



Problem 13.3 j



Problem 13.3 k

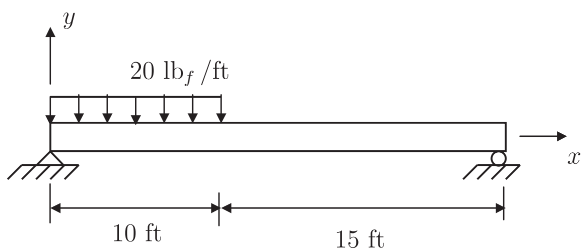


Problem 13.3 l

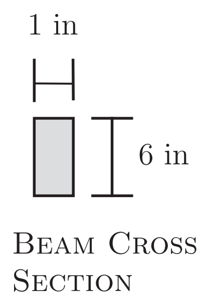
E, I are constants.

REQUIRED: (all algebra, no numbers!)

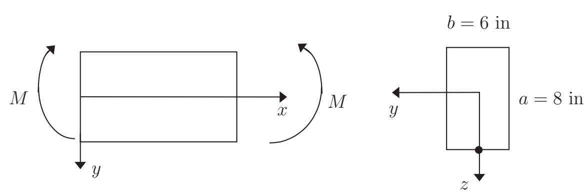
a) Draw Shear and Bending moment diagrams.



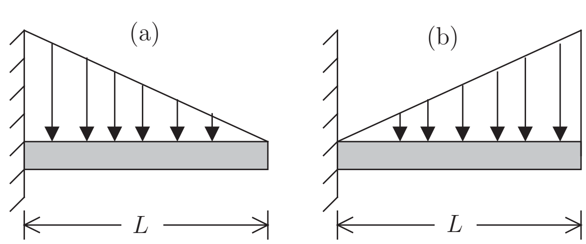
Problem 13.3 m



Problem 13.4

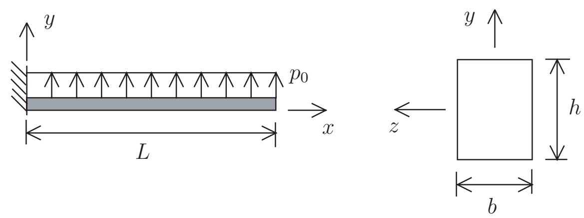


Problem 13.5



Problem 13.6

b) Assuming a rectangular cross-section, find the maximum normal tensile stress, σ_{xx}, and its location.



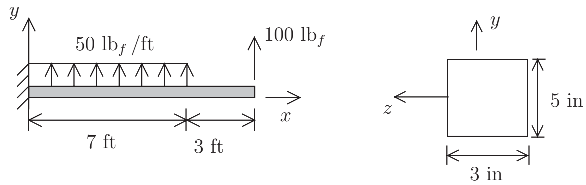
Problem 13.7

c) Determine $u(x)$, the vertical displacement at every point and graph.

d) Determine the maximum vertical displacement and its location.

13.8 *GIVEN*: The beam shown.

E, I = constants



Problem 13.8

REQUIRED:

a) Draw V & M diagrams.

b) Assuming a rectangular cross-section as shown, find the maximum normal compressive, σ_{xx}, and its location.

c) Determine the maximum shear stress and the planes on which it acts.

REQUIRED FOR 13.9 THROUGH 13.14:

a) Draw shear & moment diagrams.

b) Find the maximum normal stress, σ_{xx}, and its x location.

c) Determine the transverse displacement $u_y(x)$.

d) Determine the maximum transverse deflection and its x location.

e) Your answers may be in terms of E, I_{zz}.

13.9

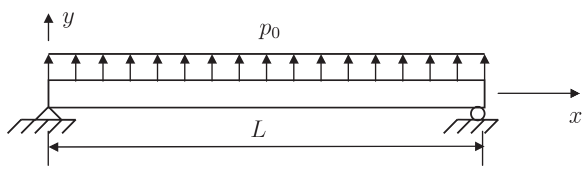
13.10

13.11

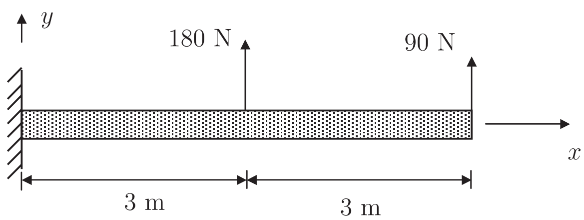
13.12

13.13

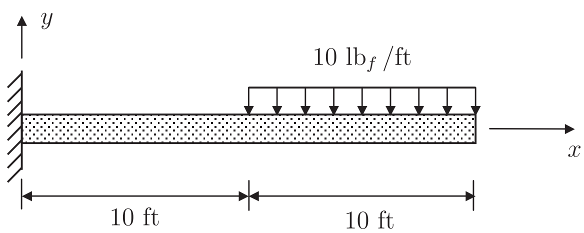
13.14



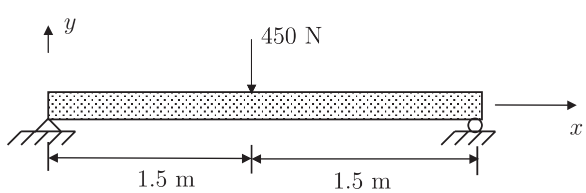
Problem 13.9



Problem 13.10



Problem 13.11



Problem 13.12

13.15 *GIVEN*: The beam shown.

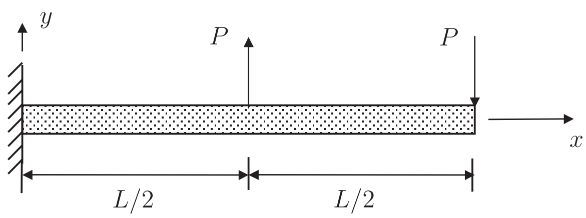
$E = 200$ GPa

$I_{zz} = 1.28 \times 10^9$ mm^4

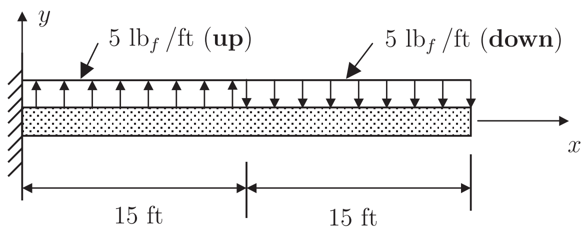
REQUIRED:

a) Draw Shear and Bending moment diagrams.

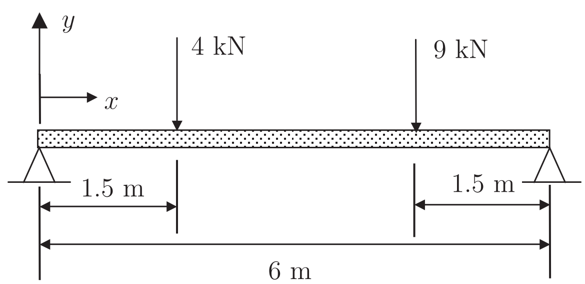
b) Assuming a rectangular cross-section, find the maximum normal tensile stress, σ_{xx}, and



Problem 13.13



Problem 13.14



Problem 13.15

its location.

c) Determine $u_y(x)$ the vertical displacement at every point and graph.

d) Determine the maximum vertical displacement and its location.

13.16 *GIVEN*: The beam shown.

$E = 2.9 \times 10^7$ psi

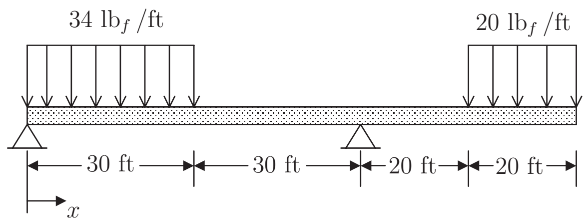
$I_{zz} = 3000$ in^4

REQUIRED:

a) Draw Shear and Bending moment diagrams.

b) Assuming a rectangular cross-section, find the maximum normal tensile stress, σ_{xx}, and its location.

c) Determine $u_y(x)$, the vertical displacement at every point and graph.

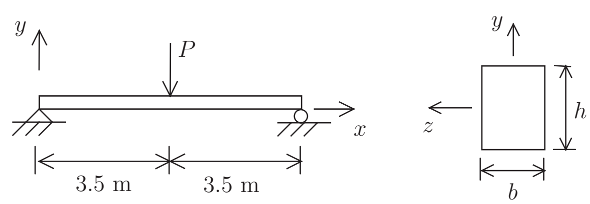


Problem 13.16

d) Determine the maximum vertical displacement and its location.

13.17 *GIVEN*: The beam shown.

$$
\begin{aligned}
P &= 45\ \frac{\text{kN}}{\text{m}} \\
L &= 7\ \text{m} \\
h &= 0.5\ \text{m} \\
b &= 0.3\ \text{m} \\
E &= 0.1029\ \text{GPa}
\end{aligned}
$$



Problem 13.17

REQUIRED:

a) Draw Shear and Bending moment diagrams.

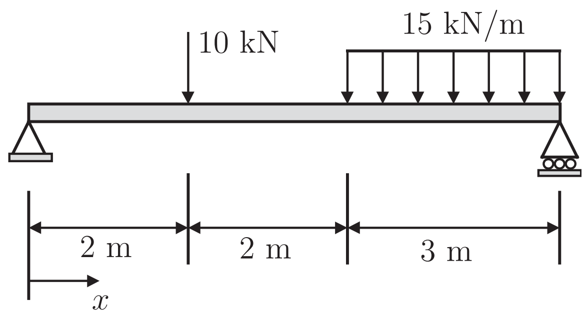
b) Assuming a rectangular cross-section, find the maximum normal tensile stress, σ_{xx}, and its location.

c) Determine $u(x)$, the vertical displacement at every point and graph.

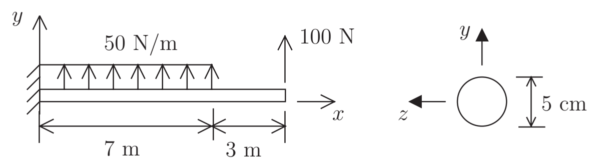
d) Determine the maximum vertical displacement and its location.

13.18 Determine the equations for the shear, $V(x)$, and moment, $M(x)$, for the beam shown below. Plot the functions along the x-axis. For a square cross sectional beam (width w = 10 cm), give the expressions for the normal stress distribution at the top of the beam. Consider the origin $(0, 0)$ to be located at the center of the cross section of the beam.

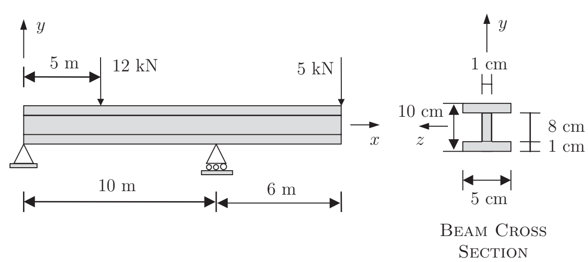
13.19 Draw Shear Force & Moment diagrams. For the given cross-section as shown, find the maximum normal compressive, σ_{xx}, and its location.



Problem 13.18



Problem 13.19



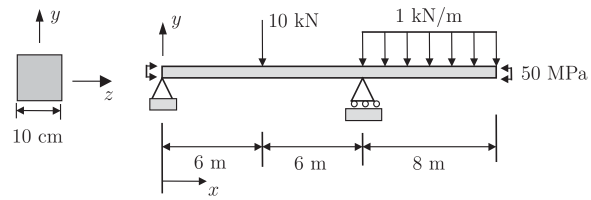
Problem 13.20

13.20 Plot the Shear Force and Moment diagrams. Compute the largest tensile and compressive bending stresses σ_{xx} in the beam shown, and show the position (x, y) where they occur. Show all calculations.

13.21 For the beam cross section shown in 13.20, calculate the moment of inertia with respect to each axis y and z (i.e. I_{yy}, I_{zz}).

13.22 Write expressions for $V(x)$ and $M(x)$ by interval for the beam shown, measuring the position of x from the location given on the left end of the bar. Considering a given compressive axial load of 50 MPa applied over the cross sectional area, calculate the position and magnitude of the maximum normal stress for a *square* cross section as shown. If the failure criteria are specified by a maximum stress of 140 MPa in tension and 250 MPa in compression, determine if the system is adequate for the applied load.

13.23 *GIVEN*: The beam below.

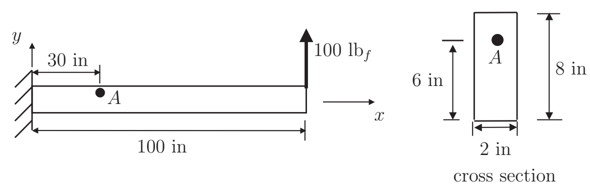


Problem 13.22

$E = 2.5 \times 10^6$ psi

Cross Section:

{ Note: Centroid is at the center of width of the cross section }



Problem 13.23

REQUIRED:

a) Find the moment of inertia, I_{zz}

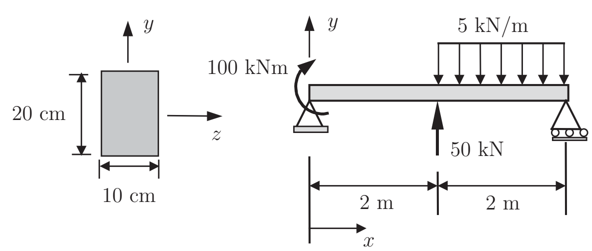
b) Compute the normal (σ_{xx}) and shear (σ_{xy}) stress at point A in the above beam.

13.24 For the beam shown below and its corresponding cross section (Note the *applied bending moment on the left end* and the concentrated 50 kN force at $x = 2$ m):

a) Obtain the expressions for shear force $V(x)$ and moment $M(x)$ along the beam.

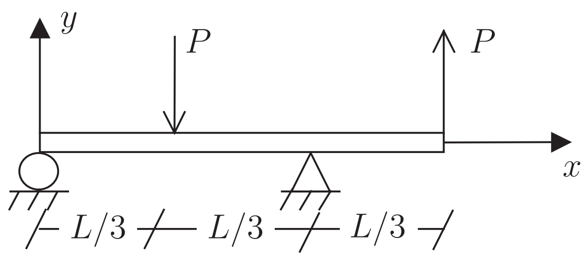
b) Draw the shear (V) and moment (M) diagrams with the appropriate units.

c) Calculate the maximum tensile stress and indicate the location where it occurs.



Problem 13.24

13.25 *GIVEN*: The simple supported beam shown below:

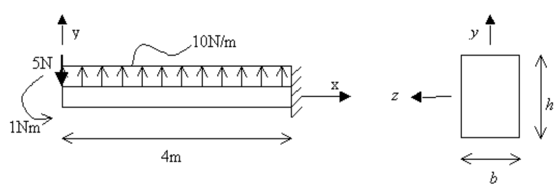


Problem 13.25

REQUIRED:

a) Draw the shear and moment diagrams by *using free-body diagrams*.
b) Determine the value and location of the maximum bending moment.

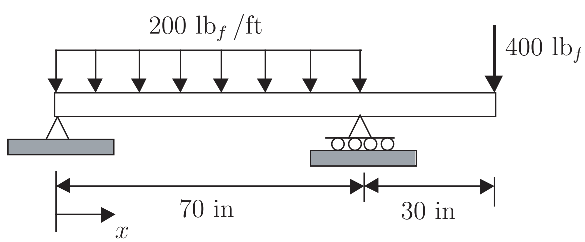
13.26 *GIVEN*: The beam shown below. E, I_{zz} = constants; $E = 10^6$ psi; $h = 2$ cm; $b = 1$ cm.



Problem 13.26

REQUIRED: Determine the shear and moment equations by the *integration method*. Plot $p_y(x)$ vs. x, $V_y(x)$ vs. x, and $M_z(x)$ vs. x.

13.27 *GIVEN*: The simple supported beam shown below.



Problem 13.27

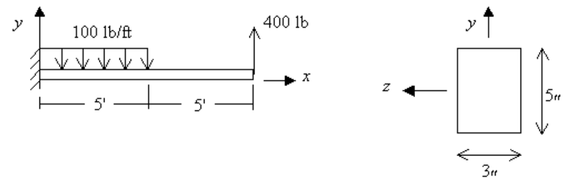
REQUIRED:

a) Use the *free-body method* to obtain the shear and moment equations as a function of x.

b) Draw shear and moment diagrams.

c) Determine the maximum value and location of the shear and bending moment.

13.28 *GIVEN*: The cantilevered beam shown below. E, I_{zz} = constants; $E = 10^6$ psi.



Problem 13.28

REQUIRED:

a) Use the *integration method* to obtain the shear and moment diagrams.

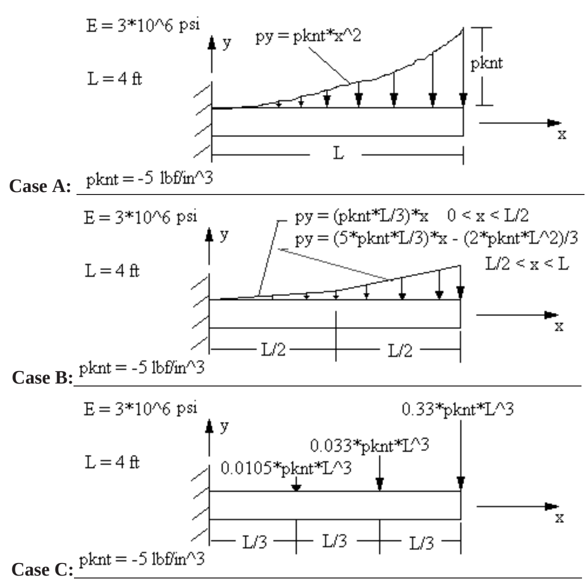
b) Draw V & M diagrams.

13.29 *GIVEN*: The three beams in Cases A, B, and C shown below.

Case A:

Case B:

Case C:



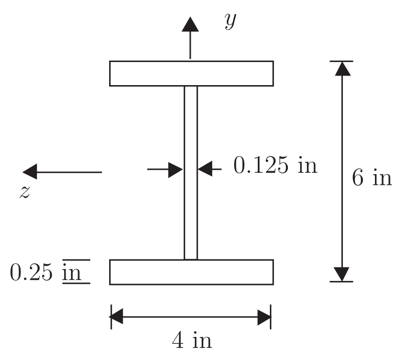
Problem 13.29

You will note that each case above is a slightly different representation of essentially the same applied load. Case A is a quadratic representation; Case B is a piecewise-linear representation and Case C is a point load representation of the applied load. We want to see how much difference these representations make in the results for shear and moment. The load is applied to the cross-section shown below:

REQUIRED:

a) Plot the loads for Case A and Case B on one set of Load vs. x axis. Plot x in feet.
b) Plot the shear diagrams for all three cases on one set of Shear vs. x axis. Plot x in feet.
c) Plot the moment diagrams for all three cases on one set of Moment vs. x axis. Plot x in feet.
d) Find the moment of inertia for the cross-section shown below about the y axis and about the z axis (units of inches).

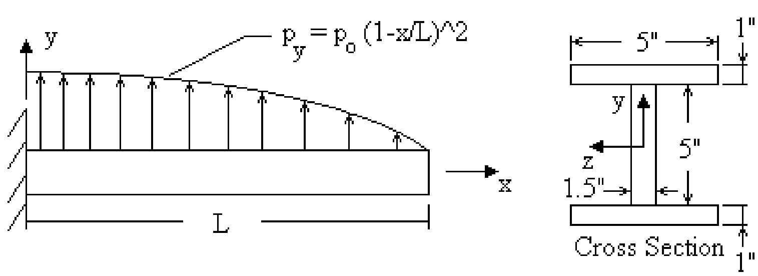
NOTE: You may want to use Scientific Workplace to produce the plots.



Problem 13.30

13.30 *GIVEN*: The cantilever beam used as a wing spar with flight loads as shown below for the General Dynamics F-16 Fighting Falcon:

E, I_{zz} = constants; E = 2024-T4 Aluminum; $L = 15.5$ ft; $p_o = 4,839\ \frac{\text{lb}_f}{\text{ft}}$.



Problem 13.31

REQUIRED:

a) Determine the shear and moment equations by the *integration method*. Plot $p_y(x)$ vs. x, $V(x)$ vs. x, $M(x)$ vs. x

b) Determine the moment of inertia of the beam about the y and z-axes.

c) Determine the deflection and stress at the bottom surface both as a function of x.

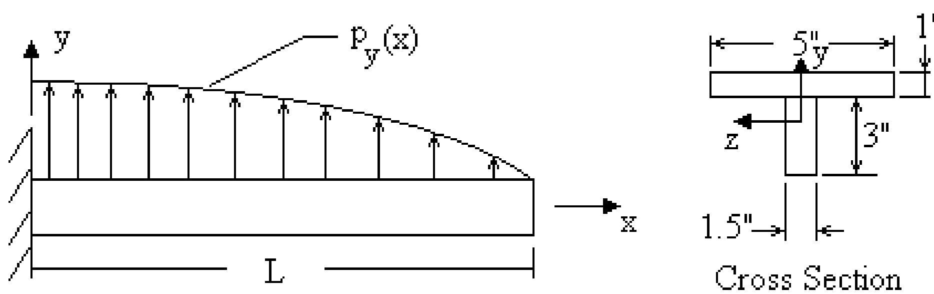
d) Plot the deflection and stress at the bottom surface as a function of x.

e) Determine the deflection at $x = L$.

f) Determine the stress at the bottom surface at $x = \frac{L}{2}$.

g) Determine the shear and bending moment at $x = 0$.

13.31 *GIVEN*: The simple supported beam shown below. E, I_{zz} = constants; E = 2024-T6 Aluminum; $L = 15.5$ ft,



Problem 13.32

Assume: $p_y(x) = p_0 \cos\left(\frac{\pi x}{2L}\right)$, $p_o = 4,839\ \frac{\text{lb}_\text{f}}{\text{ft}}$.

REQUIRED:

a) Determine the shear and moment equations by the *integration method*. Plot $p_y(x)$ vs. x, $V(x)$ vs. x, $M(x)$ vs. x

b) Determine the moment of inertia of the beam about the y and z-axes.

c) Determine the deflection and stress at the bottom surface both as a function of x.

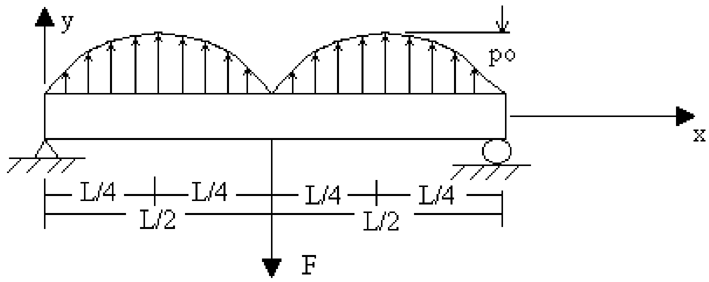
d) Plot the deflection and stress at the bottom surface as a function of x.

e) Determine the deflection at $x = L$.

f) Determine the stress at the bottom surface at $x = \frac{L}{2}$.

g) Determine the shear and bending moment at $x = 0$.

13.32 *GIVEN*: The simply supported beam shown below:



Problem 13.33

Let $L = 10$ ft, $p_o = 100\ \frac{\text{lb}_\text{f}}{\text{ft}}$, $F = 5000\ \text{lb}_\text{f}$. (note: these represents magnitudes only).

Assume the beam is made of 2024-T4 Aluminum and has a 3″ square cross-section.

$$p_y = \left[\frac{L^2}{16} - \left(x - \frac{L}{4}\right)^2\right]\frac{16p_0}{L^2} \qquad 0 \le x \le \frac{L}{2}$$

$$p_y = \left[\frac{L^2}{16} - \left(x - \frac{3L}{4}\right)^2\right]\frac{16p_0}{L^2} \qquad \frac{L}{2} \le x \le L$$

REQUIRED: Watch the units, and work the problem in lb_f, in.

a) Plot the distributed load as a function of x with x in feet.

b) Determine shear and moment equations using the integration method and plot as a function of x in ft.

c) Determine the deflection as a function of x and plot as a function of x in feet. State the location of maximum deflection.

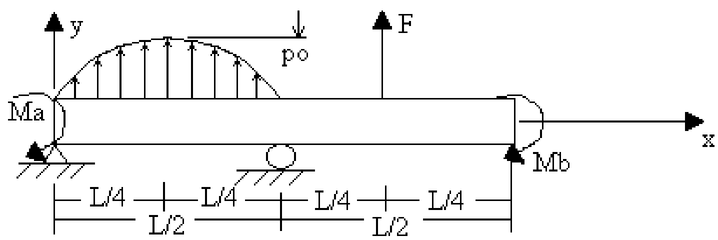
d) Determine the stress as a function of x and plot as a function of x in feet. State the location of maximum tensile stress. What is the bending moment at the location of maximum tensile stress? Is it the smallest/largest bending moment in the beam? What is the deflection at the location of maximum tensile stress?

13.33 *GIVEN*: The simply supported beam shown below:

Let $L = 16$ ft, $p_o = 1000\ \frac{\text{lb}_\text{f}}{\text{ft}}$, $F = 4000\ \text{lb}_\text{f}$, $M_a = M_b = 10,000\ \text{lb}_\text{f}$ - ft.

(note: these represents magnitudes only)

Assume: The beam is made of 2024-T4 Aluminum and has a square cross-section with a height of 2″ and base of 3″.



Problem 13.34

$$p_y = \left[\frac{L^2}{16} - \left(x - \frac{L}{4}\right)^2\right]\frac{16p_0}{L^2} \qquad 0 \leq x \leq \frac{L}{2}$$

REQUIRED: Watch the units, and work the problem in $\mathrm{lb_f}$, in.

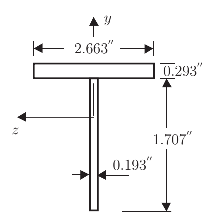
a) Plot the distributed load as a function of x with x in feet.

b) Determine shear and moment equations using the free body diagram method and plot as a function of x in ft.

c) Determine the deflection as a function of x and plot as a function of x in feet. State the location of maximum deflection.

d) Determine the stress as a function of x and plot as a function of x in feet. State the location of maximum tensile stress. What is the bending moment at the location of maximum tensile stress? Is it the smallest/largest bending moment in the beam? What is the deflection at the location of maximum tensile stress?

13.34 *GIVEN*: The AISC ST2 $\times$3.85 cross-section shown to the right.



AISC ST2×3.85

Problem 13.35

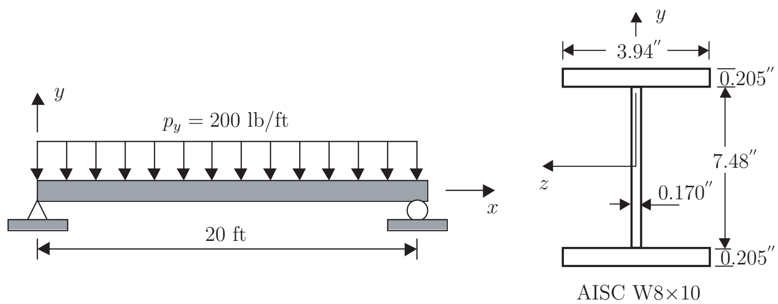
REQUIRED: Determine

a) Centroid relative to the top surface.

b) Moments of inertia about the y and z axis.

13.35 *GIVEN*: The simple supported beam below is made of A36 structural steel has an AISC W8 $\times$ 10 cross-section shown to the right.

REQUIRED: Determine:

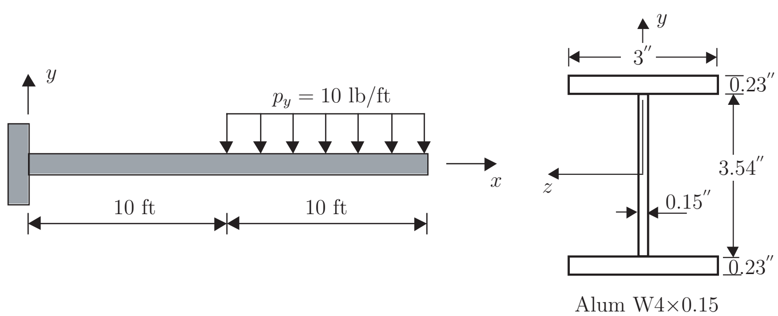


Problem 13.36

a) Determine the transverse deflection $u_y(x)$

b) The value and location of maximum bending stress (σ_{xx}) and shear stress (σ_{xy}).

13.36 *GIVEN*: The cantilevered beam shown below is made of 6061-T6 aluminum and has an Aluminum W4 × 0.15 cross-section shown to the right.



Problem 13.37

REQUIRED: Include the weight of the beam as part of the loading and determine:

a) The transverse deflection $u_y(x)$.

b) The value and location of maximum bending stress (σ_{xx}) and shear stress (σ_{xy}).

13.37 *GIVEN*: The simple supported beam below is made of A36 structural steel and has an AISC WT6 × 20 cross-section shown to the right.

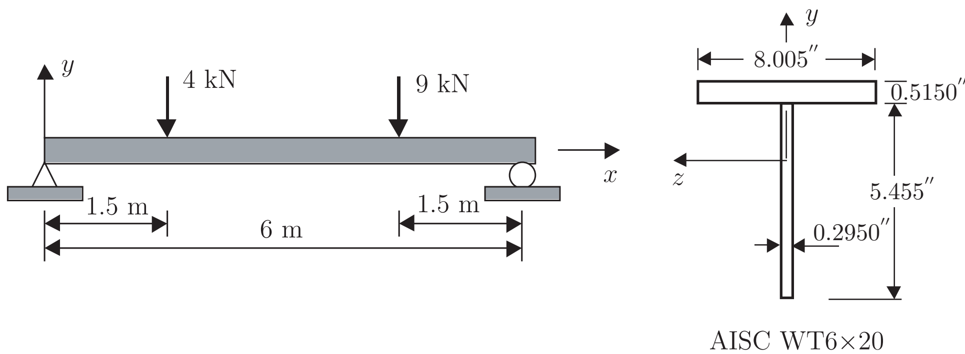
REQUIRED: Include the weight of the beam as part of the loading and determine:

a) The value and location of maximum bending stress (σ_{xx}) and shear stress (σ_{xy}).

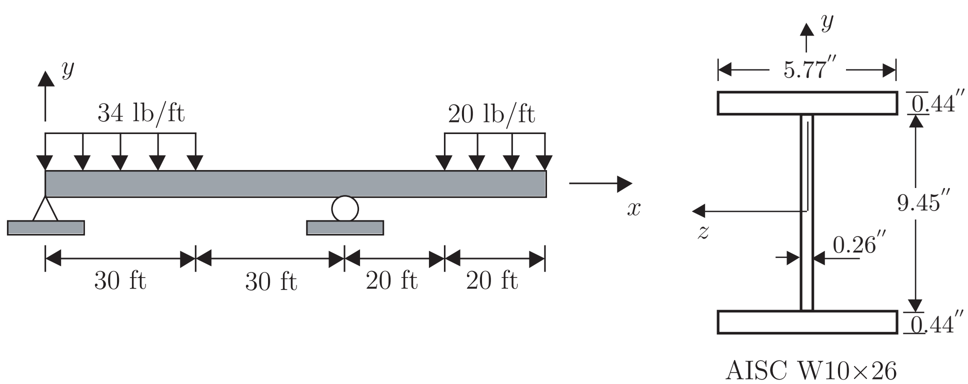
13.38 *GIVEN*: The simple supported beam below is made of A36 structural steel and has an AISC W10 × 26 cross-section shown to the right.

REQUIRED: Include the weight of the beam as part of the loading and determine:

a) The value and location of maximum bending stress (σ_{xx}) and shear stress (σ_{xy}).



Problem 13.38



Problem 13.39