Chapter 7

HEAT TRANSFER APPLICATIONS IN SOLIDS

Figure 7.1:

7.1 Problem Solving Procedure

This chapter will consider the application and solution of the heat transfer equation for a solid. Before continuing, it is instructive to introduce the problem solving method that will be used. This method includes the following components:

1. Conservation of Thermal Energy equation:

$$\rho \hat{C} \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{q} + \rho \Phi \tag{7.1}$$

2. Constitutive equation - Fourier's Law:

$$\mathbf{q} = -k \nabla T \tag{7.2}$$

3. Boundary and initial conditions for the particular problem. These may be: specified temperature or specified heat flux

It is to be noted that this solution procedure and equations (7.1) and (7.2) are valid for any coordinate system. In this chapter, problems in Cartesian and cylindrical coordinates will be considered. The general conservation of energy equation is a partial differential equation and its solution is given by $T = T(x, y, z, t)$ in Cartesian coordinates. The solution of partial differential equations is beyond the scope of this textbook. Hence, applications will be selected wherein the solution requires only the solution of an ordinary differential equation. For example, simplified cases include steady state ($\frac{\partial T}{\partial t} = 0$) and 1-D heat flow in x direction with constant k ($\nabla T = \frac{dT}{dx}$, $\nabla \cdot \mathbf{q} = -k\frac{d^2T}{dx^2}$). For the *steady-state 1-D heat flow in a solid with constant* k, the partial differential equation becomes:

$$k\frac{d^2T}{dx^2} = -\rho\Phi \tag{7.3}$$

To establish a better understanding of the conservation of thermal energy equation for solids, let us review the derivation for 1-D and 2-D. First assume the 1-D heat flow in the x-direction only through a cross-sectional area A_x, shown schematically in the figure below.

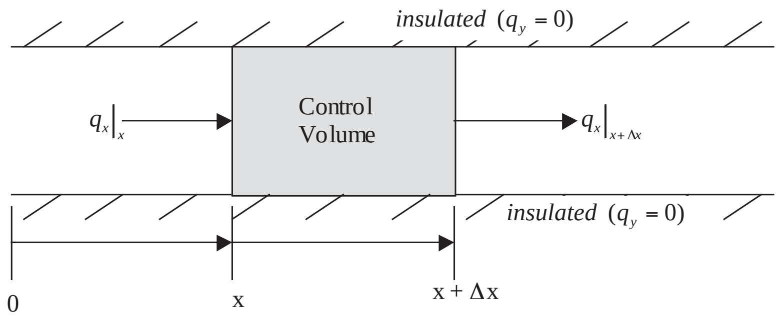


Figure 7.2: Conservation of Thermal Energy for 1-D Solid

To assume 1-D heat flow, the prismatic solid bar of cross-sectional area A_x above is insulated at its lateral surfaces so that heat flows only in the x direction. The accumulation of thermal energy in the system (infinitesimal control volume) is balanced by the net flow of thermal energy into the system:

$$\left[\left(\rho\hat{C}T\right)\Big|_{t+\Delta t} - \left(\rho\hat{C}T\right)\Big|_{t}\right] A_x\Delta x = q_x\,|_x A_x\Delta t - q_x\,|_{x+\Delta x} A_x\Delta t + \rho\Phi A_x\Delta x\Delta t \tag{7.4}$$

Assuming ρ and C independent of time, dividing by $A_x\Delta x\Delta t$ and taking the limit as $\Delta x \to 0$ and $\Delta t \to 0$, we have from the above equation:

$$\rho\hat{C}\frac{\partial T}{\partial t} = -\frac{\partial q_x}{\partial x} + \rho\Phi \tag{7.5}$$

Note that there are two unknown quantities, $T(x, t)$ and $q_x(x, t)$ in the above equation. The conservation of thermal energy for a solid in 2-D can similarly be derived by assuming zero heat flow in the z-direction. Assume the thickness of the slab is L in the z direction.

$$\begin{aligned}\left[\left(\rho\hat{C}T\right)\Big|_{t+\Delta t} - \left(\rho\hat{C}T\right)\Big|_{t}\right]\Delta x\Delta yL &= \left(q_x|_x - q_x|_{x+\Delta x}\right)\Delta yL\Delta t \\ &+ \left(q_y|_y - q_y|_{y+\Delta y}\right)\Delta xL\Delta t + \rho\Phi\Delta x\Delta yL\Delta t\end{aligned} \tag{7.6}$$

Dividing by $\Delta x\Delta yL\Delta t$ and taking the limit as Δx, Δz and $\Delta t \to 0$, one obtains the conservation of thermal energy for a solid in 2-D:

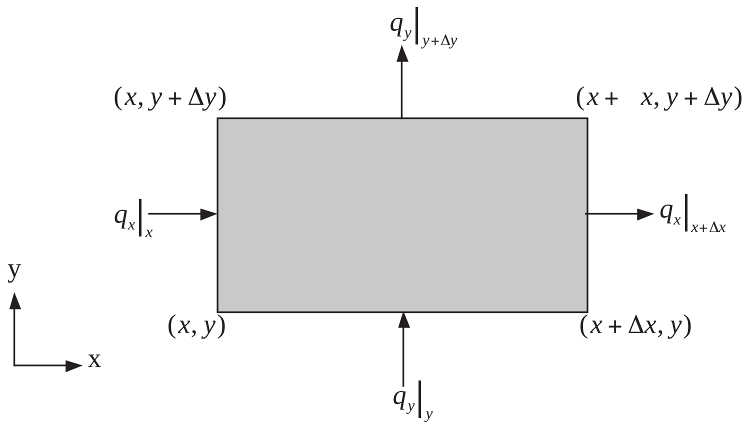


Figure 7.3: Conservation of Thermal Energy for 2-D Solid

$$\rho \hat{C} \frac{\partial T}{\partial t} = -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} + \rho \Phi \tag{7.7}$$

The above differential equation has $T(x, y, t)$, $q_x(x, y)$, and $q_y(x, y, t)$ as unknown functions, assuming that ρ, C and Φ are given. Similarly, the 3-D conservation of thermal energy equation can be obtained with an explicit evaluation in Cartesian coordinates

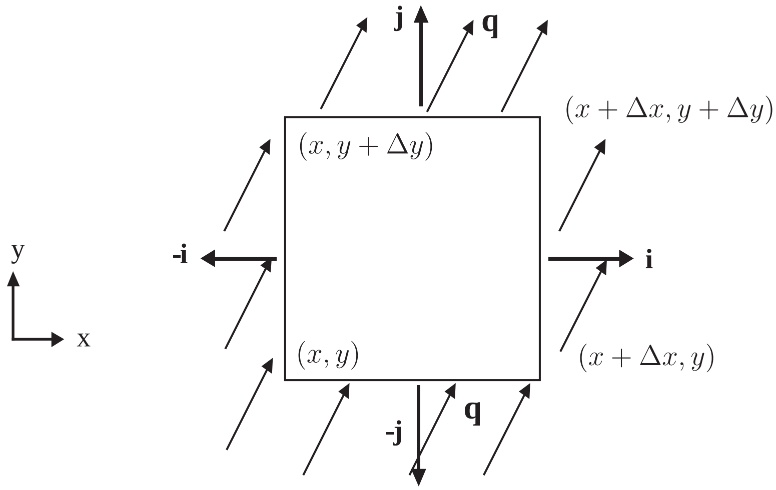
$$\rho \hat{C} \frac{\partial T}{\partial t} = -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z} + \rho \Phi \tag{7.8}$$

or the vector form shown in Equation (7.1). Note that now there are four unknown functions, $T(x, y, z, t)$, $q_x(x, y, z, t)$, $q_y(x, y, z, t)$, $q_z(x, y, z, t)$ to be found from only one equation. There are no additional conservation laws available to use to determine the unknown functions. Before further discussion, we will redraw the control volume element of Figure (7.3) in order to make clear the reasoning behind the heat flux components.

To reduce the number of unknowns in equation 7.7 we have introduced Fourier's law of heat conduction, which has the form:

$$\begin{aligned}
&\text{(1-D):} \qquad q_x = -k \frac{\partial T(x, t)}{\partial x} \\
&\text{(2-D):} \qquad q_x = -k \frac{\partial T(x, y, t)}{\partial x}, \quad q_y = -k \frac{\partial T(x, y, t)}{\partial y} \\
&\text{(3-D):} \qquad q_x = -k \frac{\partial T(x, y, z, t)}{\partial x}, \quad q_y = -k \frac{\partial T(x, y, z, t)}{\partial y}, \quad q_z = -k \frac{\partial T(x, y, z, t)}{\partial z}
\end{aligned} \tag{7.9}$$

The quantity k is a material property called the coefficient of thermal conduction. When Fourier's Law is substituted into equations (7.5), (7.7), and (7.8), the conservation of thermal energy for solids reduces to:



Heat flux through x-face leaving the system: $\mathbf{q} \cdot \mathbf{i} = q_x|_{x+\Delta x}$
Heat flux through $-x$-face leaving the system: $\mathbf{q} \cdot (-\mathbf{i}) = -\, q_x|_x$
Heat flux through y-face leaving the system: $\mathbf{q} \cdot (-\mathbf{j}) = -\, q_y|_y$
Heat flux through $-y$-face leaving the system: $\mathbf{q} \cdot (\mathbf{j}) = q_y|_{y+\Delta y}$

Figure 7.4: Infinitesimal Control Volume Element for 2-D Heat Transfer

$$\begin{aligned} &\text{(1-D):} \qquad \rho \hat{C} \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + \rho \Phi \\ &\text{(2-D):} \qquad \rho \hat{C} \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} + \rho \Phi \\ &\text{(3-D):} \qquad \rho \hat{C} \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} + k \frac{\partial^2 T}{\partial z^2} + \rho \Phi \end{aligned} \tag{7.10}$$

where k is assumed to be constant. In each equation (7.10) there is only one unknown function $T(x, t)$ for 1-D, $T(x, y, t)$ for 2-D, and $T(x, y, z, t)$ for 3-D. The material properties ρ and C, and the heat source term Φ are all assumed to be known functions of x, y, z and t.

7.2 Various Cases of the Heat Transfer Equation

Here we compile the various cases of the conservation of energy equation for a solid with Fourier's Law incorporated, i.e., the heat transfer equation.

1. General 3-D case (with $k = k(x, y, z, t)$):

$$\rho \hat{C} \frac{\partial T}{\partial t} = -\nabla(-k \nabla T) + \rho \Phi \tag{7.11}$$

2. Thermal conductivity, k, is constant:

$$\rho\hat{C}\frac{\partial T}{\partial t} = k\nabla^2 T + \rho\Phi \tag{7.12}$$

$$\rho\hat{C}\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\rho\Phi, \qquad T = T(\,x,y,z,t\,)$$

3. Steady-State (where the temperature is constant with time) and constant k:

$$k\nabla^2 T = -\rho\Phi T = T(\,x,y,z\,) \tag{7.13}$$

4. 1-D Steady-State with constant k:

$$k\frac{d^2T}{dx^2}(x) = -\rho\Phi T = T(x) \tag{7.14}$$

7.3 Boundary Conditions

In order to complete the solution of any differential equation, appropriate boundary conditions must be specified. In heat transfer through a solid, there are three types of boundary conditions that will be considered herein.

1. Specified surface temperature: $T(\,x,y,z,t\,) = T_S(\,x,y,z,t\,)$, where the subscript "$S$" refers to the boundary surface of the body.

2. Specified heat flux on surface with normal $\mathbf{n}$: $\mathbf{n}\cdot\mathbf{q} = \mathbf{n}\cdot(-k\nabla T) = function(x,y,z,t)$. The special case of an insulated boundary where there is no heat flux is given by: $\mathbf{n}\cdot\mathbf{q} = 0$.

3. Specified heat flux due to *convective heat transfer* on the boundary: $\mathbf{n}\cdot\mathbf{q} = h(T_S - T_\infty)$ where

   - T_S = surface temperature (***unknown***)
   - T_∞ = far-field temperature (***specified/known***)
   - h = heat convection coefficient (*from experiment*)

Note: h has units of $\frac{\mathrm{J}}{(\mathrm{m}^2\ \mathrm{s}\ ^\circ\mathrm{C})} = \frac{\mathrm{W}}{(\mathrm{m}^{2\,\circ}\mathrm{C})}$.

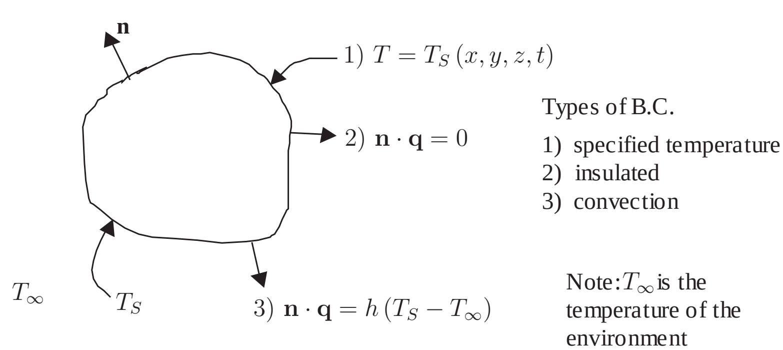


Figure 7.5: Solid with Boundary Conditions Shown

Another type of temperature boundary condition is radiation to the environment from a body. The radiative boundary condition takes the form of $\mathbf{q}\cdot\mathbf{n} = \kappa(T_S^4 - T_\infty^4)$ where κ is a radiation constant. This type of boundary condition is important but will not be considered here since it leads to nonlinear equations.

Consider two bodies that are touching so that they have an interface between them as shown below. Assume that the temperature distribution in bodies 1 and 2 are $T_1(x, y, z, t)$ and $T_2(x, y, z, t)$, respectively. Thermal conductivity for the two bodies is k_1 and k_2. *At the interface of two solids*,

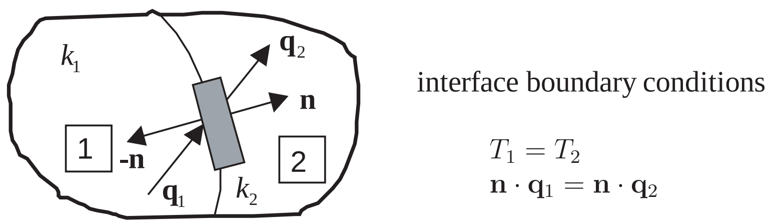


Figure 7.6: Two Bodies in Contact

there are two "boundary" conditions that must be met:

1. The temperature of each body at the interface must be equal: $T_1 = T_2$.
2. The heat flux leaving body 1 at the interface must equal the heat flux entering body 2, i.e., the heat flux is constant across the interface: $\mathbf{n}_1 \cdot \mathbf{q}_1 = \mathbf{n}_2 \cdot \mathbf{q}_2$. For a 1-D case with heat flow only in the x direction, this reduces to $k_1 \frac{\partial T_1}{\partial x} = k_2 \frac{\partial T_2}{\partial x}$ where the subscripts refer to the thermal conductivity and temperature gradient in bodies 1 and 2, respectively.

The use of symbols for various material properties and temperature provides very useful information in terms of general solutions as well as being able to see how the various terms combine and contribute to the final solution. Examples given below will include numerical values of material properties such as thermal conductivity k and convection coefficient h. Some typical values are listed below.

Material	$k\ \left[\frac{\mathrm{J}}{\mathrm{m\text{-}s\text{-}^\circ C}}\right]$	*Material*	$k\ \left[\frac{\mathrm{J}}{\mathrm{m\text{-}s\text{-}^\circ C}}\right]$
Silver	428	Water	0.6
Copper	398	Soil	0.52
Aluminum 2024-T3	190	Polyethylene	0.38
Aluminum 6061-T6	156	Teflon	0.25
Nickel	89.9	Nylon	0.24
Iron	80.4	Polystyrene	0.13
Magnesia	37.7	Polypropylene	0.12
Alumina	30.1	White Pine	0.11
Steel AISI 304	16.3	Glass wool	0.04
Spinel	15.0	Polyurethane foam	0.026
Titanium B 120VCA	7.4		
Ice	2.2		
Concrete	1.8		
Glass	1.7		
Soil (42% water)	1.1		

Fluid	$h\ \left[\frac{\mathrm{W}}{(\mathrm{m}^2\ \mathrm{K})}\right]$
Air (free convection)	5-25
Air (forced convection)	25-250

Useful units and conversions
1 W = $\frac{\text{J}}{\text{s}}$ = 3.41 $\frac{\text{BTU}}{\text{hr}}$, 1 HP = 550 $\frac{\text{ft-lb}}{\text{s}}$ = 746 W, 1 m = 3.281 ft = 39.37 in,
1 $\frac{\text{W}}{(\text{m}^2\ \text{K})}$ = 0.176 $\frac{\text{BTU}}{(\text{hr ft}^2\,{}^\circ\text{R})}$, 1 $\frac{\text{W}}{(\text{m K})}$ = 0.578 $\frac{\text{BTU}}{(\text{hr ft }{}^\circ\text{R})}$, 1 BTU = 1055 J,
K = $\frac{5}{9}$ °R = °C + 273 = $\frac{5}{9}$ (460 + °F), °C = $\frac{5}{9}$ (°F − 32)

An example of converting h is given below:

$$h = 5\frac{\text{BTU}}{\text{hr}\cdot\text{ft}^2\,{}^\circ\text{F}} = 5\frac{1055\text{ J}}{(3600\text{ s})(0.3048\text{ m})^2\frac{5}{9}\,{}^\circ\text{C}} = 5\times 5.678\frac{\text{J}}{\text{m}^2\text{s}\cdot{}^\circ\text{C}} = 28.39\left[\frac{\text{J}}{\text{m}^2\text{s}\cdot{}^\circ\text{C}}\right]$$

7.4 Selected Applications

Example 7-1

Steady-state heat conduction along an insulated bar

Consider steady-state heat conduction in a bar of length L that is insulated along its lateral sides. The material has a thermal conductivity of k. The temperature at the left end ($x = 0$) is T_0 and at $x = L$, $T(x = L) = T_L$. Determine the temperature distribution, $T = T(x)$.

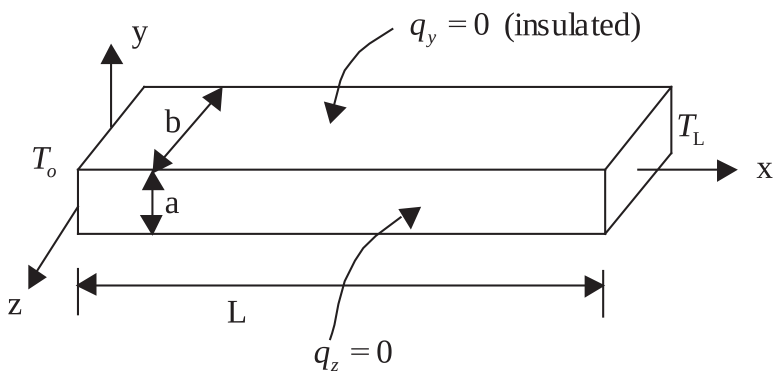


Figure 7.7: Insulated Bar with Temperature Boundary Conditions

- Solve the Partial Differential Equation for the 1-D case:

$$\nabla^2 T = \frac{d^2T}{dx^2} = 0 \implies T(x) = C_1 + C_2 x, \quad \mathbf{q} = -k\frac{dT}{dx}\mathbf{i} = -kC_2\mathbf{i}$$

  Note that we have two unknown constants of integration C_1 and C_2. Thus two boundary conditions are required.

- Satisfy Boundary Conditions on lateral surfaces:

$$\mathbf{n}\cdot\mathbf{q} = 0 \implies \mathbf{n}\cdot(\nabla T) = 0 \text{ is satisfied for the lateral surfaces.}$$

- Satisfy Boundary Conditions at the ends:

$$T(0) = T_0 = C_1$$
$$T(L) = C_1 + C_2 L = T_L \implies T_0 + C_2 L = T_L \implies C_2 = \frac{T_L - T_0}{L}$$

Substituting C_1 and C_2 into $T(x)$, the temperature distribution along the length of bar is given by:

$$T = T(x) = T_0 + \frac{T_L - T_0}{L} x$$

Example 7-2

Steady-state heat conduction through a slab

Consider steady-state heat conduction through a slab (wall) of thickness L that convects heat to the environment on both sides as shown below. The material has a thermal conductivity of k. The

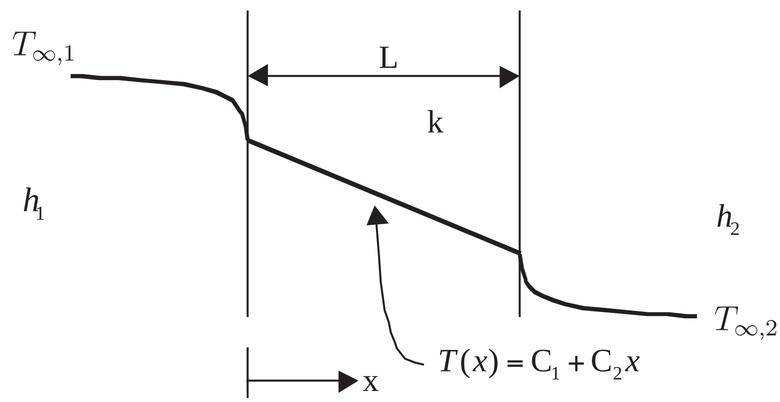


Figure 7.8: Wall with Convection Boundary Conditions

temperature of the environment is $T_{\infty,1}$ (left side) and $T_{\infty,2}$ (right side). The convection coefficient on the left side of the wall is h_1 and on the right side h_2.

a) Determine the temperature distribution, $T = T(x)$, through the slab.

b) Determine the heat flux in the slab.

c) For a slab that has an area A though which heat is flowing (heat flows in the x direction through the area A with unit normal $\mathbf{i}$), determine the total amount of heat energy passing through the slab in a time period Δt.

- Solve the Partial Differential Equation for the 1-D case:

$$\nabla^2 T = \frac{d^2 T}{dx^2} = 0 \implies T(x) = C_1 + C_2 x, \quad \mathbf{q} = -k \frac{dT}{dx} \mathbf{i} = -k C_2 \mathbf{i}$$

  Note that we have two unknown constants of integration C_1 and C_2. Thus two boundary conditions are required.

- Satisfy Boundary Conditions on each side of the wall:
  - Left side of wall at $x = 0$: $\mathbf{n} = -1\mathbf{i}$ (convection boundary condition):

$$\begin{aligned} \mathbf{n} \cdot \mathbf{q}\big|_{x=0} &= h(T_S - T_\infty)\big|_{x=0} \\ -\mathbf{i} \cdot \left(-k \frac{dT}{dx} \mathbf{i} \right)\bigg|_{x=0} &= kC_2 = h_1 \left((C_1 + C_2(0)) - T_{\infty,1} \right) \\ kC_2 &= h_1 \left(C_1 - T_{\infty,1} \right) \;\; \ldots\ldots\ldots\ldots\ldots\ldots \;\; (1) \end{aligned}$$

    Note that what we have done is equate the heat flux in the slab and the heat flux in the left environment at the left boundary ($x = 0$).

- Right side of wall at $x = L$: $\mathbf{n} = +1\mathbf{i}$ (convection boundary condition):

$$\begin{aligned}
\mathbf{n}\cdot\mathbf{q}|_{x=L} &= h(T_S - T_\infty)|_{x=L} \\
+\mathbf{i}\cdot\left(-k\frac{dT}{dx}\mathbf{i}\right)\bigg|_{x=0} &= -kC_2 = h_2\left((C_1 + C_2(L)) - T_{\infty,2}\right) \\
-kC_2 &= h_2\left(C_1 + C_2L - T_{\infty,2}\right) \ \ldots\ldots\ldots\ldots\ldots \ (2)
\end{aligned}$$

Equations (1) and (2) above may now be solved for the constants of integration. We have a system of two equations:

$$\begin{bmatrix} h_1 & -k \\ h_2 & (k + h_2L) \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} h_1T_{\infty,1} \\ h_2T_{\infty,2} \end{Bmatrix} \tag{7.15}$$

Solving equations (7.15) gives

$$\begin{aligned}
C_1 &= \frac{k(h_1T_{\infty,1} + h_2T_{\infty,2}) + h_1h_2LT_{\infty,1}}{kh_1 + h_1h_2L + kh_2} \\
C_2 &= \frac{h_1h_2(T_{\infty,2} - T_{\infty,1})}{kh_1 + h_1h_2L + kh_2}
\end{aligned} \tag{7.16}$$

The constants of integration (7.16) may be substituted into the general solution for $T(x)$ to obtain the temperature distribution through the wall:

$$T(x) = C_1 + C_2x = \frac{k(h_1T_{\infty,1} + h_2T_{\infty,2}) + h_1h_2LT_{\infty,1}}{kh_1 + h_1h_2L + kh_2} + \frac{h_1h_2(T_{\infty,2} - T_{\infty,1})}{kh_1 + h_1h_2L + kh_2}x$$

The heat flux through the wall is given by Fourier's Law:

$$\mathbf{q} = -k\frac{dT}{dx}\mathbf{i} = -kC_2\mathbf{i} = -k\frac{h_1h_2(T_{\infty,2} - T_{\infty,1})}{kh_1 + h_1h_2L + kh_2}\mathbf{i} \tag{7.17}$$

The last result may be written in a slightly different, more informative, way. Divide the numerator and denominator of right side of equation (7.17) by kh_1h_2 to obtain

$$-\frac{kh_1h_2(T_{\infty,2} - T_{\infty,1})}{kh_1 + h_1h_2L + kh_2}\,\frac{\frac{1}{kh_1h_2}}{\frac{1}{kh_1h_2}} = -\frac{(T_{\infty,2} - T_{\infty,1})}{\frac{1}{h_2} + \frac{L}{k} + \frac{1}{h_1}} = -\frac{T_{\infty,2} - T_{\infty,1}}{R}$$

where $R \equiv \frac{1}{h_2} + \frac{L}{k} + \frac{1}{h_1}$ so that heat flux is given by:

$$\mathbf{q} = -\frac{(T_{\infty,2} - T_{\infty,1})}{R}\mathbf{i} \tag{7.18}$$

The quantity R *is called the effective thermal resistance* and will be discussed later in this chapter. Note that if $T_{\infty,1}$ is higher than $T_{\infty,2}$ (as shown on the sketch above), then equation (7.18) (or (7.17)) indicates that heat flow is to the right as expected.

The total heat flow energy passing through a wall with an area A in time Δt is obtained by multiplying the heat flux by the area and time:

$$Q = q_xA\Delta t \tag{7.19}$$

Note: The solution for the system of equations (7.15) may be obtained using Scientific Workplace:

$h_1C_1 - kC_2 = h_1T_{\infty,1}$
$h_2C_1 + (k + h_2L)C_2 = h_2T_{\infty,2}$,

Solution is : $\left\{C_1 = \frac{h_1kT_{\infty,1} + h_1h_2LT_{\infty,1} + h_2T_{\infty,2}k}{h_2k + h_1k + h_1h_2L}, C_2 = -h_1h_2\frac{-T_{\infty,2} + T_{\infty,1}}{h_2k + h_1k + h_1h_2L}\right\}$

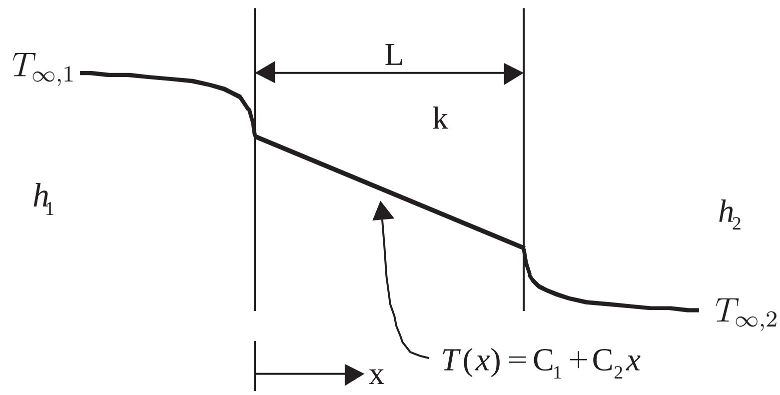


Figure 7.9: Wall with Convection Boundary Conditions

Example 7-3

Steady-state heat conduction through a slab (numerical example)

This example is identical to the previous example, except with numerical values for material constants and the boundary conditions. Consider steady-state heat conduction through a slab (wall) of thickness 2-cm that convects heat to the environment on both sides as shown below.

The material is aluminum and has a thermal conductivity of 247 $\frac{\text{J}}{(\text{m s C})}$. The temperature of the environment is 50 °C (left side) and 20 °C (right side). The convection coefficient on the left side of wall is 40 $\frac{\text{W}}{(\text{m}^2\,{}^\circ\text{C})}$ and on the right side is 10 $\frac{\text{W}}{(\text{m}^2\,{}^\circ\text{C})}$.

a) Determine the temperature distribution, $T = T(x)$, through the slab.

b) Determine the temperature at the left and right boundaries.

c) Determine the heat flux in the slab.

d) For a slab that has an area 2 m^2 though which heat is flowing (heat flows in the x direction through the area A with unit normal $\mathbf{i}$), determine the total amount of heat energy passing through the slab in 1 hour.

- Solve the Partial Differential Equation for the 1-D case:

$$\nabla^2 T = \frac{d^2 T}{dx^2} = 0 \implies T(x) = C_1 + C_2 x, \quad \mathbf{q} = -k\frac{dT}{dx}\mathbf{i} = -kC_2\mathbf{i}$$

  Note that we have two unknown constants of integration C_1 and C_2. Thus two boundary conditions are required.

- Satisfy Boundary Conditions on each side of the wall

  – Left side of wall at $x = 0$: $\mathbf{n} = -1\mathbf{i}$ (convection boundary condition):

$$\begin{aligned}
\mathbf{n}\cdot\mathbf{q}\big|_{x=0} &= h(T_S - T_\infty)\big|_{x=0} \\
-\mathbf{i}\cdot\left(-k\frac{dT}{dx}\mathbf{i}\right)\bigg|_{x=0} &= kC_2 = h_1\left((C_1 + C_2(0)) - T_{\infty,1}\right) \\
kC_2 = h_1\left(C_1 - T_{\infty,1}\right) &\implies \\
247\,\frac{\text{J}}{\text{m - s - C}}C_2 &= 40\,\frac{\text{J}}{\text{m}^2\text{ - s - C}}\left(C_1 - 50\ {}^\circ\text{C}\right)\ \ .....\ (1)
\end{aligned}$$

Note that what we have done is equate the heat flux in the slab and the heat flux in the left environment at the left boundary ($x = 0$).

– Right side of wall at $x = L$: $\mathbf{n} = +1\mathbf{i}$ (convection boundary condition):

$$\begin{aligned}
\mathbf{n}\cdot\mathbf{q}\big|_{x=L} &= h(T_S - T_\infty)\big|_{x=L} \\
+\mathbf{i}\cdot\left(-k\frac{dT}{dx}\mathbf{i}\right)\bigg|_{x=0} &= -k(C_2) = h_2\left((C_1 + C_2(L)) - T_{\infty,2}\right) \\
-kC_2 = h_2\left(C_1 + C_2L - T_{\infty,2}\right) &\Longrightarrow \\
-247\ \frac{\text{J}}{\text{m - s - °C}}C_2 &= 10\ \frac{\text{J}}{\text{m}^2\text{ - s - °C}}\left(C_1 + C_2(.02\text{ m}) - 20\text{ °C}\right)\quad ..\ (2)
\end{aligned}$$

Equations (1) and (2) directly above may now be solved for the constants of integration. We have a system of two equations:

$$\begin{bmatrix} 40 & -247 \\ 10 & 247.2 \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 2000 \\ 200 \end{Bmatrix} \tag{7.20}$$

Solving equations (7.20) gives

$$\begin{aligned}
C_1 &= 44.0\text{ °C} \\
C_2 &= -0.971\left(\frac{\text{°C}}{\text{m}}\right)
\end{aligned}$$

Substituting C_1 and C_2 into $T(x)$, the temperature distribution through the wall is given by:

$$T(x) = C_1 + C_2x = 44 - 0.971x\text{ °C} \tag{7.21}$$

The temperature at any point in the slab may now be obtained by substituting an x position into equation (7.21). At the left boundary $x = 0$ and at the right boundary $x = 0.02$ m so that

$$\begin{aligned}
\text{left boundary}: &\quad T(x = 0) = 44 - 0.971(0.0) = 44\text{ °C} \\
\text{right boundary}: &\quad T(x = 0.02) = 44 - 0.971(0.02) = 43.98\text{ °C}
\end{aligned}$$

The last result shows that aluminum is not a good insulator since the temperature on both sides of the wall is practically identical (the thermal conductivity k for aluminum is relatively large). To a person on the right side of the wall where the environmental temperature is 20 °C, the wall would feel very hot to the touch!

The heat flux through the wall is given by

$$\mathbf{q} = -k\frac{dT}{dx}\mathbf{i} = -kC_2\mathbf{i} = -247\ \frac{\text{J}}{(\text{m - s - °C})}\left(-0.971\ \frac{\text{°C}}{\text{m}}\right)\mathbf{i} = +239.8\ \left(\frac{\text{J}}{\text{m}^2\text{ - s}}\right)\mathbf{i}$$

We could have determined the effective thermal resistance R

$$R \equiv \frac{1}{h_2} + \frac{L}{k} + \frac{1}{h_1} = \frac{1}{10\ \frac{\text{J}}{\text{m}^2\text{ - s - °C}}} + \frac{.02\text{ m}}{247\ \frac{\text{J}}{\text{m - s - C}}} + \frac{1}{40\ \frac{\text{J}}{\text{m}^2\text{ - s - °C}}} = 0.12508\ \frac{\text{m}^2\text{ - s - °C}}{\text{J}}$$

so that heat flux is given by: $\mathbf{q} = -\frac{(T_{\infty,2} - T_{\infty,1})}{R}\mathbf{i} = -\frac{(20\text{ °C} - 50\text{ °C})}{.12508\frac{\text{m}^2\text{ - s - °C}}{\text{J}}}\mathbf{i} = 239.8\mathbf{i}\frac{\text{J}}{\text{m}^2\text{ - s}}$

Note that since $T_{\infty,1}$ on the left is higher than $T_{\infty,2}$ on the right, then the heat flow is positive and to the right as expected.

The total heat flow energy flowing through an area of 2 m^2 in 1 hour is obtained by multiplying the heat flux by the area and time:

$$Q = q_xA\Delta t = 239.8\ \frac{\text{J}}{\text{m}^2\text{ - s}}(2\text{ m}^2)\left(1\text{ hr}\frac{3600\text{ s}}{\text{hr}}\right) = 1.73\times 10^6\text{ J}$$

Example 7-4

The wall thickness of a refrigerator must be designed to maintain the temperature shown below (given a $-5\ ^\circ$C *wall* temperature on the inside of the refrigerator and 40 °C *environmental* temperature on the outside of the refrigerator) with the additional requirement that *the heat flux cannot exceed* $1 \times 10^2\ \frac{\mathrm{J}}{(\mathrm{m}^2\text{ -s})}$.

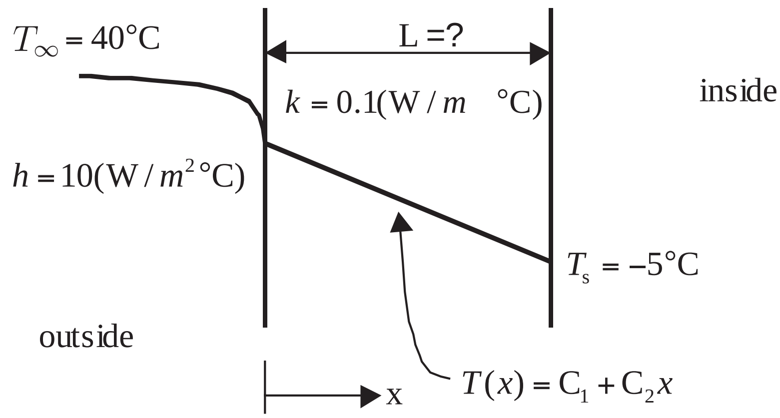


Figure 7.10: Refrigerator Wall

The wall is constructed of a foam insulating material with thermal conductivity of $0.1\ \frac{\mathrm{J}}{(\mathrm{m\ s\ ^\circ C})}$. The refrigerator is a rectangular box with a total surface wall area of 4 m^2.

a) What is the minimum required wall thickness?

b) What is the outside wall temperature?

c) What is the temperature gradient across the wall?

d) How many kilowatts of energy are lost per day due to convection?

- Solve the Partial Differential Equation for the 1-D case:

$$\nabla^2 T = \frac{d^2T}{dx^2} = 0 \implies T(x) = C_1 + C_2 x, \quad \mathbf{q} = -k\frac{dT}{dx}\mathbf{i} = -kC_2\mathbf{i}$$

  Note that we have two unknown constants of integration C_1 and C_2 but we also have a third unknown, which is the wall thickness L (total of 3 unknowns). We have two boundary conditions on temperature (convection on left side and specified temperature on right wall) plus the third condition for the specified heat flux.

- Satisfy Boundary Conditions on each side of the wall:
  - Left side of wall at $x = 0$: $\mathbf{n} = -1\mathbf{i}$ (convection boundary condition):

$$\begin{aligned}
\mathbf{n}\cdot\mathbf{q}\big|_{x=0} &= h(T_S - T_\infty)\big|_{x=0} \\
-\mathbf{i}\cdot\left(-k\frac{dT}{dx}\mathbf{i}\right)\Big|_{x=0} &= kC_2 = h\left((C_1 + C_2(0)) - T_\infty\right) \\
kC_2 = h\left(C_1 - T_\infty\right) &\implies \\
0.1\frac{\mathrm{J}}{\mathrm{m\ \text{-}\ s\ \text{-}\ ^\circ C}}C_2 &= 10\frac{\mathrm{J}}{\mathrm{m^2\ \text{-}\ s\ \text{-}\ ^\circ C}}\left(C_1 - 40\ ^\circ\mathrm{C}\right) \quad .... \ (1)
\end{aligned}$$

– Right side of wall at $x = L$: $\mathbf{n} = +1\mathbf{i}$ and $T = -5$ °C (specified temperature):

$$T(x = L) = C_1 + C_2(L) = -5 \text{ °C} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

- Notice that at this point, we have two equations, (1) and (2), but three unknowns: C_1, C_2 and L. The third equation is obtained from the heat flux requirement.

$$q_x = -k\frac{dT}{dx} = -kC_2 = -0.1C_2\frac{\text{W}}{\text{m °C}} \leq 1 \times 10^2 \ \frac{\text{W}}{\text{m}^2}$$
$$C_2 \leq -1 \times 10^3 \left(\frac{\text{°C}}{\text{m}}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

We will take $C_2 = -10^3 \left(\frac{\text{°C}}{\text{m}}\right)$ (i.e., the largest value that still satisfies (3)). A smaller value would produce a smaller heat flux than the allowed requirement but would produce a larger thickness.

Equation (1), (2) and (3) directly above may be solved for the three unknowns to obtain.

$$\begin{aligned} C_2 &= -1000 \left(\frac{\text{°C}}{\text{m}}\right) \\ C_1 &= 30 \text{ °C} \\ L &= 0.035 \text{ m} \end{aligned}$$

Hence the minimum wall thickness is $L = 3.5$ cm. Any wall thickness greater than this will produce a smaller heat flux than the specified allowable maximum.

b) The temperature distribution through the wall is given by:

$$T(x) = C_1 + C_2 x = 30 - 10^3 x \text{ °C}$$

The outside wall of the refrigerator is at $x = 0$ and hence we have

$$\text{outside wall temperature: } T(x = 0) = 30 - 10^3(0.0) = 30 \text{ °C}$$

c) The temperature gradient through the wall is simply $\frac{dT}{dx}$:

$$\text{temperature gradient in wall} = \frac{dT}{dx} = C_2 = -10^3 \ \frac{\text{°C}}{\text{m}}$$

The magnitude of the temperature gradient is large which shows that the foam is a very good insulator (has a small thermal conductivity k). In this case, the outside wall temperature is 30 °C while the inside wall temperature is -5 °C with a wall thickness of only 2.5 cm.

7.5 Heat Conduction Through a Composite Flat Wall

Consider two plane walls in contact (called a composite wall) as shown below. The individual walls are labeled 1 and 2 as are each the thermal conductivity k and thickness L. Assume the wall boundaries convect heat to the environment on both sides. Each side may have different convection coefficients h and environmental temperature T_{inf}.

Each layer must satisfy the heat conduction equation $\nabla^2 T = \frac{\partial^2 T}{\partial x^2} = 0$ whose solution is a linear function in x. Consequently, we have the following solution for layers 1 and 2:

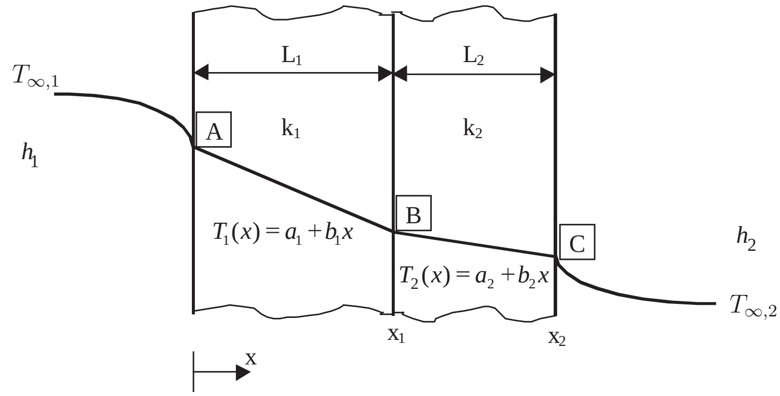


Figure 7.11: Composite Wall With Convection Boundary Conditions

$$\begin{aligned} \nabla^2 T_1 &= \frac{d^2 T_1}{dx^2} = 0 \quad \Longrightarrow \quad T_1(x) = a_1 + b_1 x \\ \nabla^2 T_2 &= \frac{d^2 T_2}{dx^2} = 0 \quad \Longrightarrow \quad T_2(x) = a_2 + b_2 x \end{aligned} \tag{7.22}$$

To evaluate the constants a_1, a_2, b_1, and b_2, the boundary conditions at points A and C and the interface conditions at B must be satisfied.

A) Convective boundary condition at $x = 0$: (note direction of $\mathbf{n}$, $\mathbf{n} = -\mathbf{i}$)

$$\begin{aligned} &\mathbf{q}\cdot\mathbf{n}|_{x=0} = h\left(T_S - T_\infty\right)|_{x=0} \\ &\Longrightarrow \left(-k_1 \frac{dT_1}{dx}\mathbf{i}\right)\cdot(-\mathbf{i}) = k_1 \left.\frac{dT_1}{dx}\right|_{x=0} = k_1 b_1 = h_1\left(T_1 - T_{\infty,1}\right)|_{x=0} = h_1\left(a_1 - T_{\infty,1}\right) \\ &\Longrightarrow k_1 b_1 = h_1\left(a_1 - T_{\infty,1}\right) \end{aligned} \tag{7.23}$$

B) Interface boundary condition between wall 1 and 2 at $x = x_1$:

$$T_1(x_1) = T_2(x_1) \Longrightarrow a_1 + b_1 x_1 = a_2 + b_2 x_1 \tag{7.24}$$

$$(\mathbf{q}\cdot\mathbf{n})_{\text{slab 1}} = (\mathbf{q}\cdot\mathbf{n})_{\text{slab 2}} \Longrightarrow \left(-k_1 \frac{dT_1}{dx}\mathbf{i}\right)\cdot\mathbf{i} = \left(-k_2 \frac{dT_2}{dx}\mathbf{i}\right)\cdot\mathbf{i} \Longrightarrow b_1 k_1 = b_2 k_2 \tag{7.25}$$

C) Convective boundary condition at $x = x_2$: (note change in $\mathbf{n}$ direction, $\mathbf{n} = +\mathbf{i}$)

$$\begin{aligned} &\mathbf{q}\cdot\mathbf{n}|_{x=x_2} = h\left(T_S - T_\infty\right)|_{x=x_2} \\ &\Longrightarrow \left(-k_2 \frac{dT_2}{dx}\mathbf{i}\right)\cdot(+\mathbf{i}) = -k_2 \left.\frac{dT_2}{dx}\right|_{x=x_2} = -k_2 b_2 = h_2\left(T_2 - T_{\infty,2}\right)|_{x=x_2} = h_2\left(a_2 + b_2 x_2 - T_{\infty,2}\right) \\ &\Longrightarrow -k_2 b_2 = h_2\left(a_2 + b_2 x_2 - T_{\infty,2}\right) \end{aligned} \tag{7.26}$$

Consequently we have four equations and four unknowns (a_1, a_2, b_1, and b_2) as follows:

$$\begin{aligned} k_1 b_1 &= h_1 (a_1 - T_{\infty,1}) \\ a_1 + b_1 x_1 &= a_2 + b_2 x_1 \\ b_1 k_1 &= b_2 k_2 \\ k_2 b_2 &= -h_2 (a_2 + b_2 x_2 - T_{\infty,2}) \end{aligned} \tag{7.27}$$

The above system of four equations can be solved for a_1, a_2, b_1, and b_2 and the result substituted back into equation (7.22) to obtain $T_1(x)$ and $T_2(x)$. The heat flux in the x direction is given by

$$q_x = -h_1 (a_1 - T_{\infty,1}) = k_1 b_1 = k_2 b_2 = h_2 ((a_2 + b_2 x_2) - T_{\infty,2}) \tag{7.28}$$

For 1-D slab heat flow, heat can flow only in one direction (in this case, the x direction). Consequently, in the absence of heat sinks/sources in a layer, the heat flux must remain a constant as it passes through the convective air layer on the left, through each slab and finally through the convective air layer on the right.

To simplify the solution for a composite wall (*with no internal heat source*), we seek to develop a simplified relation between the overall heat flux through the composite wall and the given temperature gradient from one side of the composite wall to the other:

$$\begin{aligned} q_x = -U\Delta T \quad \text{or} \quad q_x &= -\left(\frac{1}{R}\right)\Delta T \\ \Delta T &= T_{\infty,2} - T_{\infty,1} \end{aligned} \tag{7.29}$$

where $U \equiv$ effective heat transfer coefficient of the composite wall, $R = \frac{1}{U} \equiv$ ***effective thermal resistance*** of the composite wall and, for the case of convection boundary conditions on each side of the composite wall, the known temperature gradient from left to right is given by $\Delta T = T_{\infty,2} - T_{\infty,1}$.

The solution of the ODE for heat transfer through a single layer with no heat source requires that the temperature variation in the layer is a linear function of x: $T(x) = a + bx$ where a and b are constants of integration dependent on boundary conditions. If the temperature on either side of a wall of thickness L is T_A and T_B, then $T(x) = T_A + \left[\frac{(T_B - T_A)}{L}\right] x$. For the composite wall shown below, we introduce the following notation. Define the temperature at the left boundary to be T_A, at the interface T_B, and at the right boundary T_C as shown in the figure below. At this point, T_A, T_B, and T_C are unknown.

In the absence of a heat source within the body, the temperature in each layer will be a linear function of x so that we may write the following equations for the temperature in each layer:

$$\begin{aligned} T_1(x) &= T_A + \left(\frac{T_B - T_A}{L_1}\right) x \\ T_2(x) &= T_B + \left(\frac{T_C - T_B}{L_2}\right)(x - x_1) \end{aligned} \tag{7.30}$$

At $x = L_1$ (the interface), these equations already satisfy the interface condition that $T_1(x) = T_2(x) = T_B$. Therefore, only the heat flux boundary condition needs to be satisfied at the interface and the convective boundary condition at the left and right boundaries of the composite wall. From conservation of energy, the heat energy Q_x through a given area A ($Q_x = q_x A$) must be constant as it enters on the left and leaves on the right boundary (since we assumed there is no internal heat generation, Φ). Since heat flow is normal to wall, each layer has same normal area (so area cancels

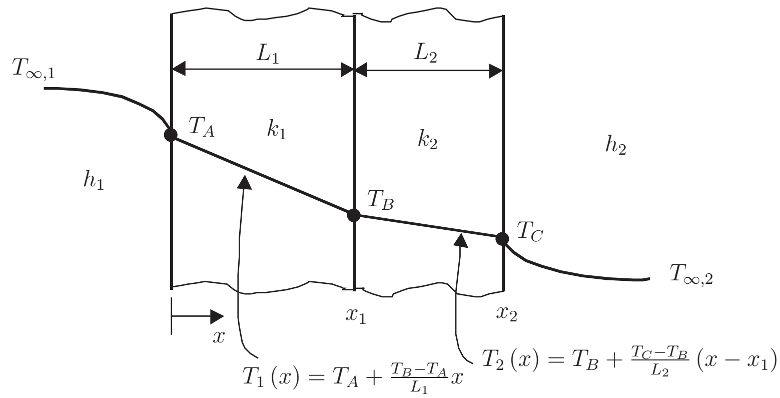


Figure 7.12: Composite Wall With Convective Boundary Conditions

out). Thus, the heat flux q_x must remain a constant as it passes through the convective air layer on the left, through each slab and finally through the convective air layer on the right and we can write

$$q_x = -h_1 (T_A - T_{\infty,1}) = -k_1 \frac{T_B - T_A}{L_1} = -k_2 \frac{T_C - T_B}{L_2} = h_2 (T_C - T_{\infty,2}) \tag{7.31}$$

Equation (7.31) may be separated into 4 equations by considering each heat flux term individually to obtain:

$$\begin{array}{rcll} T_{\infty,1} - T_A & = & \dfrac{1}{h_1} q_x & (1) \\ T_B - T_C & = & \dfrac{L_2}{k_2} q_x & (2) \\ T_A - T_B & = & \dfrac{L_1}{k_1} q_x & (3) \\ T_C - T_{\infty,2} & = & \dfrac{1}{h_2} q_x & (4) \end{array} \tag{7.32}$$

Add these four equations, (1) through (4) to obtain

$$T_{\infty,1} - T_{\infty,2} = +\left(\frac{1}{h_1} + \frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{1}{h_2} \right) q_x \tag{7.33}$$

or,

$$q_x = -(T_{\infty,2} - T_{\infty,1}) \quad \frac{1}{\underbrace{\frac{1}{h_1} + \frac{L_1}{k_1} + \frac{L_2}{k_2} + \frac{1}{h_2}}_{\substack{\text{thermal resistance of} \\ \text{the composite wall}}}} \tag{7.34}$$

The fractional term in (7.34) may be defined as the **effective heat transfer coefficient** U:

$$U = \text{effective heat transfer coefficient} = \frac{1}{\frac{1}{h_1} + \sum\limits_{i=1}^{n} \frac{L_i}{k_i} + \frac{1}{h_2}} \tag{7.35}$$

where n is the number of layers in the composite wall. We may also define **the effective thermal resistance** R by the reciprocal of U:

$$R = \text{effective thermal resistance} = \frac{1}{U} = \frac{1}{h_1} + \sum_{i=1}^{n} \frac{L_i}{k_i} + \frac{1}{h_2} \tag{7.36}$$

Consequently, the heat flux q_x through the composite *wall with convection boundary conditions on both sides of the wall* is given by

$$\begin{aligned} q_x &= -U\Delta T = -\frac{\Delta T}{R} \\ \text{where} \quad \Delta T &= T_{\infty,2} - T_{\infty,1} \end{aligned} \tag{7.37}$$

Note that thermal resistance terms (like $\frac{L}{k}$ or $\frac{1}{h}$) are additive similar to resistors in electrical theory.

The last result may be expanded to include various boundary conditions on the left and right side of the composite wall. For example, for a composite wall with 3 layers we obtain the following summary of results:

Summary of Conduction Through Composite Walls

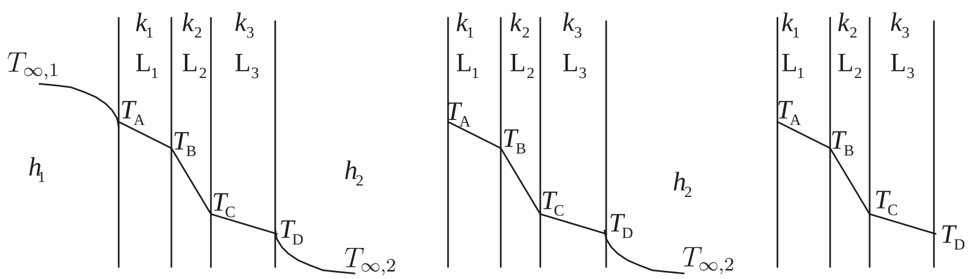


Figure 7.13:

$$\begin{array}{lll} R = \dfrac{1}{h_1} + \displaystyle\sum_{i=1}^{N} \frac{L_i}{k_i} + \frac{1}{h_2} & R = \displaystyle\sum_{i=1}^{N} \frac{L_i}{k_i} + \frac{1}{h_2} & R = \displaystyle\sum_{i=1}^{N} \frac{L_i}{k_i} \\ q_x = -\left(T_{\infty,2} - T_{\infty,1}\right)\dfrac{1}{R} & q_x = -\left(T_{\infty,2} - T_A\right)\dfrac{1}{R} & q_x = -\left(T_D - T_A\right)\dfrac{1}{R} \end{array} \tag{7.38}$$

where the heat flux in each layer is given by:

$$\begin{aligned} q_x &= -h_1\left(T_A - T_{\infty,1}\right) \\ q_x &= -\frac{k_1}{L_2}\left(T_B - T_A\right) \\ q_x &= -\frac{k_2}{L_2}\left(T_C - T_B\right) \\ q_x &= -\frac{k_3}{L_3}\left(T_D - T_C\right) \\ q_x &= -h_2\left(T_{\infty,2} - T_D\right) \end{aligned} \tag{7.39}$$

Note: the fist and last terms below represent heat flux through the fluid layers where convection occurs (terms with h) and the 2^{nd} through 4^{th} terms represent heat flux through the solid layers where conduction occurs (terms with k).

Considering the definition of R (7.38) for the various cases of different boundary conditions we note that when there is convection on the left and right, the terms h_1 and h_2 appear in R. When there is convection only on the right, only h_2 appears, etc. For three solid layers, we have $\frac{L}{k}$ for each of the three layers. This suggests the following simplified definition of R:

$$R = \text{effective thermal resistance} = \frac{1}{U} = \left\langle \frac{1}{h_1} \right\rangle + \sum_{i=1}^{n} \frac{L_i}{k_i} + \left\langle \frac{1}{h_2} \right\rangle \tag{7.40}$$

where $\langle\rangle$ means to include the h term only if there is convection on left (h_1) or right (h_2)

The general solution procedure then consists of three steps:

1. Evaluate effective thermal resistance R using (7.40)
2. Evaluate the heat flux q_x for the composite wall using $q_x = -\left(\frac{1}{R}\right)\Delta T$ where ΔT = (right most temperature) – (left most temperature).
3. Evaluate the temperatures for each layer using (7.39) working from left to right through the layers. T_A can be obtained from the first equation in (7.39), T_B from second equation, etc.

Example 7-5

Consider a two-layer composite wall with 1-D heat transfer through the layers and free convection of air on either side with $h = 5\,\frac{\text{BTU}}{(\text{hr ft}^2\ {}^\circ\text{F})}$. Assume the thickness of each layer is $L_1 = L_2 = 10$ cm. The temperature difference from left to right is $(T_{\infty,2} - T_{\infty,1}) = 50\ {}^\circ\text{C}$. Find q_x for the following situations:

a) Material 1-glass; Material 2-glass

b) Material 1-copper; Material 2-glass

c) Material 1-copper; Material 2-teflon

Solution

a) h is first converted to metric:

$$h = 5\,\frac{\text{BTU}}{\text{h}\cdot\text{ft}^2\ {}^\circ\text{F}} = 5 \times \frac{1055\ \text{J}}{(3600\ \text{s})(0.3048\ \text{m})^2 \frac{5}{9}\ {}^\circ\text{C}} = 5 \times 5.68 \frac{\text{J}}{\text{m}^2\ \text{s}\cdot{}^\circ\text{C}} = 28.39 \left[\frac{\text{J}}{\text{m}^2\ \text{s}\cdot{}^\circ\text{C}}\right]$$

$$\begin{aligned} R &= \frac{1}{28.39} + \frac{0.1}{1.7} + \frac{0.1}{1.7} + \frac{1}{28.39} = (0.035 + 0.058)\,2 = 0.188\,\frac{\text{m}^2\ {}^\circ\text{C}}{\text{W}} \\ q_x &= -50\frac{1}{R} = -50\frac{1}{0.188} = -265.8 \left[\frac{\text{W}}{\text{m}^2}\right] \qquad \left[\begin{array}{l}\text{free convection} \\ \text{glass/glass}\end{array}\right] \end{aligned}$$

b)

$$\begin{aligned} R &= \frac{1}{28.39} + \frac{0.1}{398} + \frac{0.1}{1.7} + \frac{1}{28.39} = 0.1295 \\ q_x &= -50\frac{1}{R} = -386 \left[\frac{\text{W}}{\text{m}^2}\right] \qquad \left[\begin{array}{l}\text{free convection} \\ \text{copper/glass}\end{array}\right] \end{aligned}$$

c)

$$\begin{aligned} R &= 2\frac{1}{28.39} + \frac{0.1}{328} + \frac{0.1}{0.25} = 0.47 \\ q_x &= -50\frac{1}{R} = -106.2 \left[\frac{\text{W}}{\text{m}^2}\right] \qquad \left[\begin{array}{l}\text{free convection} \\ \text{copper/teflon}\end{array}\right] \end{aligned}$$

Note that in case c), the introduction of teflon, which is a good insulator with a relatively low coefficient of thermal conductivity k, yields a higher effective resistance R and correspondingly lower heat flux, q_x.

Example 7-6

Consider a two layer composite wall of copper and teflon as shown below. The copper has a thickness of 10 cm but the thickness of the teflon is to be determined. The temperature on the left boundary is equal to 200 °C and on the right boundary 25 °C. Determine the thickness of the teflon layer so that the heat flux is equal to 200 $\frac{\text{W}}{\text{m}^2}$.

Given:

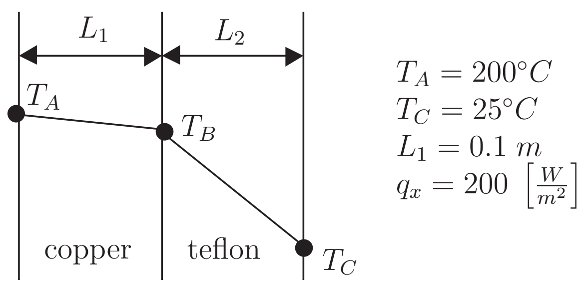


Figure 7.14:

Find: L_2

Solution

$$\begin{aligned} q_x &= -U\Delta T = -\frac{1}{R}\Delta T \\ R &= -\frac{\Delta T}{q_x} = -\frac{25-200}{200} = 0.875\ \frac{°\text{C - m}^2}{\text{W}} \\ R &= \frac{L_1}{k_1} + \frac{L_2}{k_2} = \frac{0.1\text{ m}}{398\ \frac{\text{J}}{\text{m - s}}} + \frac{L_2}{0.25\ \frac{\text{J}}{\text{m - s}}} = 0.875\ \frac{°\text{C - m}^2}{\text{W}} \implies L_2 = 0.22\text{ m} \end{aligned} \tag{7.41}$$

Example 7-7

Consider steady-state heat conduction through a cylindrical wall with convection on both sides of the cylindrical wall. Find the temperature of the wall.

The heat transfer equation in cylindrical coordinates is given by

$$\left.\begin{array}{l}\nabla^2 T = 0 \\ T(r_1)\end{array}\right\}, \quad \frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0 \implies T = C_1 \ln r + C_2 \tag{7.42}$$

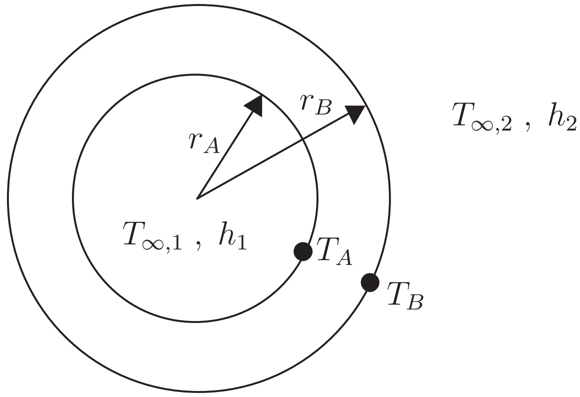


Figure 7.15:

In the absence of internal a heat source in the solid, the solution provided above will always hold.
at A

$$\mathbf{q}\cdot\mathbf{n} = h_1\left(T_A - T_{\infty,1}\right) \implies -k\frac{dT}{dr}\mathbf{e}_r\cdot(-\mathbf{e}_r) = k\frac{dT}{dr} = h_1\left(T_A - T_{\infty,1}\right) \tag{7.43}$$

at B

$$\mathbf{q}\cdot\mathbf{n} = h_2\left(T_B - T_{\infty,2}\right) \implies -k\frac{dT}{dr}\mathbf{e}_r\cdot\mathbf{e}_r = -k\frac{dT}{dr} = h_2\left(T_B - T_{\infty,2}\right) \tag{7.44}$$

Substituting (7.42) into the previous two boundary condition equations yields:

$$\begin{aligned} kC_1\frac{1}{r_A} &= h_1\left(C_1\ln r_A + C_2 - T_{\infty,1}\right) \\ kC_1\frac{1}{r_B} &= -h_2\left(C_1\ln r_B + C_2 - T_{\infty,2}\right) \end{aligned} \tag{7.45}$$

Equations (7.45) may be solved for C_1 and C_2 and substituted into (7.42) to obtain the solution for the temperature distribution $T(r)$.

Example 7-8

Consider steady-state heat conduction through a cylindrical wall with specified temperature on the boundaries of the cylindrical wall. Find the temperature of the wall.

The solution of the heat flow equation in cylindrical coordinates is given by

$$\left.\begin{array}{l}\nabla^2 T = 0 \\ T(r_1)\end{array}\right\}, \quad \frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0 \implies T(r) = C_1\ln r + C_2$$

Applying the boundary conditions at the inner and outer radius gives

$$\begin{aligned} T(r_i) = T_i &= C_1\ln r_i + C_2 \;\;.......\;\; (1) \\ T(r_o) = T_o &= C_1\ln r_o + C_2 \;\;.......\;\; (2) \end{aligned}$$

Subtracting equation (2) from equation (1) gives:

$$T_i - T_o = C_1\ln\left(\frac{r_i}{r_o}\right)$$

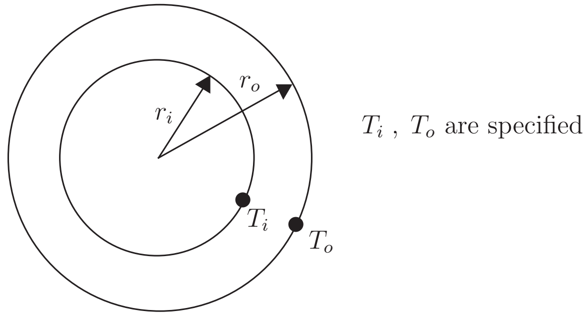


Figure 7.16:

or,

$$C_1 = \frac{T_i - T_o}{\ln\left(\frac{r_i}{r_o}\right)}$$

Substituting C_1 into equation (1) above gives

$$C_2 = T_i - \frac{T_i - T_o}{\ln\left(\frac{r_i}{r_o}\right)} \ln(r_o)$$

Substituting C_1 and C_2 into $T(r)$ yields

$$T = \frac{T_i - T_o}{\ln\left(\frac{r_i}{r_o}\right)} \ln r + \left(T_i - \frac{(T_i - T_o)\ln r_i}{\ln\left(\frac{r_i}{r_o}\right)} \right) = T_i + (T_i - T_o)\frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_i}{r_o}\right)}$$

or

$$T(r) = T_i - (T_i - T_o)\frac{\ln\left(\frac{r}{r_i}\right)}{\ln\left(\frac{r_o}{r_i}\right)}$$

The heat flux in the radial direction is given by:

$$q_r(r) = -k\frac{dT}{dr} = +k\left((T_i - T_o)\frac{1}{r}\frac{1}{\ln\left(\frac{r_o}{r_i}\right)} \right)$$

or

$$q_r(r) = k\frac{1}{r}\frac{T_i - T_o}{\ln\left(\frac{r_o}{r_i}\right)}$$

Note that the heat flux q_r is a function of radial position r. This is necessary because the area through which the heat flows increases as r increases. The radial flow Q_r for time Δt is given by $Q_r = q_r A\Delta t = q_r(2\pi r)\Delta t = 2\pi\Delta t k\frac{T_i - T_o}{\ln\left(\frac{r_0}{r_i}\right)}$. Note that $Q_r$is independent of r (as it should be) since there is no internal heat source and thus the heat flow must be the same at all radial positions, r.

Deep Thought

Comfort, like heat, can be conducted through the human touch.

7.6 Questions

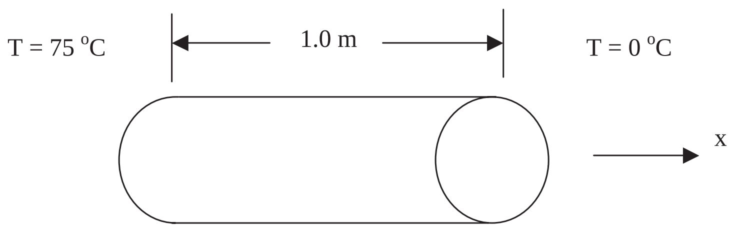
7.1 What are the three types of boundary conditions and their corresponding equations that are used frequently in relation to heat transfer?

7.2 List and explain the heat conduction problem solving method.

7.3 What is the equation used when solving a problem about conduction through a cylindrical wall? What kind of equation is this, and how many boundary conditions are required to solve it?

7.7 Problems

7.4 *GIVEN*: A laterally insulated rod, as shown below, with a uniform heat source of $\rho\Phi = 1.0 \frac{\text{J}}{(\text{m}^3\ \text{sec})}$. Heat flows only in the x direction.



Problem 7.4

REQUIRED: Calculate the temperature field $T(x)$ inside the rod for two different thermal conductivity coefficients:

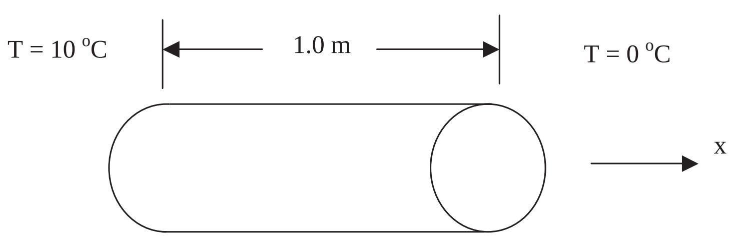
(1) $k_{\text{Copper}} = 398 \frac{\text{J}}{(\text{sec m } ^\circ\text{K})}$

(2) $k_{\text{Nylon}} = 0.24 \frac{\text{J}}{(\text{sec m } ^\circ\text{K})}$

(3) Show graphically the temperature distribution $T(x)$ for these two cases using the same scale.

(4) Calculate the heat flux (heat transfer rate) to the surroundings at either end.

7.5 *GIVEN*: A laterally insulated rod without any heat source inside the rod, as shown below. Heat flows only in the x direction.



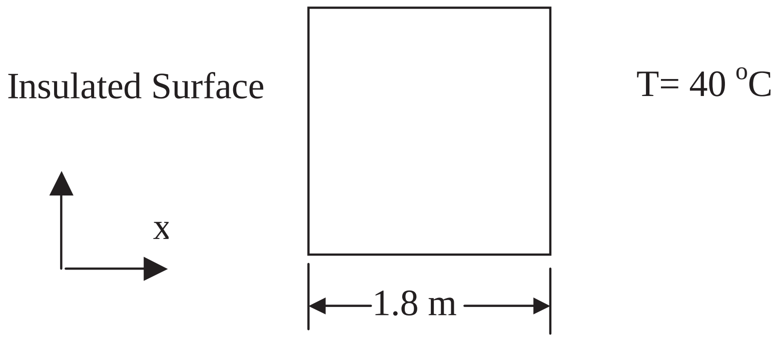
Problem 7.5

REQUIRED: For two different materials: (1) Copper, and (2) Nylon, using the data given in 7.4,

(1) Are the temperature distributions inside the rod for these two different materials the same? Why?

(2) Compute the heat transfer rate from the left surface of the rod to the right surface of the rod. Are they the same? Why?

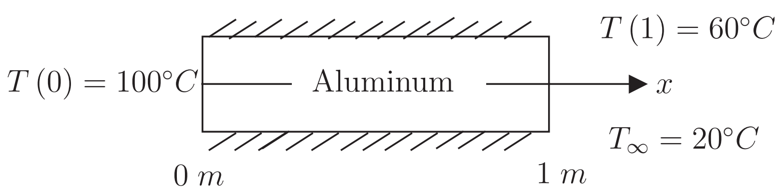
7.6 *GIVEN*: A slab, as shown below, with heat source $\rho\Phi = 4x \frac{\mathrm{J}}{(\mathrm{m}^3\ \mathrm{sec})}$, and thermal conductivity $k = 2.0 \frac{\mathrm{J}}{(\mathrm{sec\ m}\ ^\circ\mathrm{K})}$. Heat flows only in the x direction.



Problem 7.6

REQUIRED: Determine the temperature field $T = T(x)$. Draw the curve T vs. x.

7.8 *GIVEN*: Consider an insulated rod with heat flow in the x direction only. At the left boundary, the temperature is 100 °C. On the right boundary, convection occurs and the following is known: temperature at right boundary is 60 °C and the environmental temperature is 20 °C. The convection coefficient is unknown.



Problem 7.8

a) Find q_x.

b) Find h.

7.9 Consider a two layer slab with heat flow through the slab. The following material properties are known:

Material	$k\left[\frac{\mathrm{J}}{\mathrm{m - s - °C}}\right]$		
Aluminum	247		
Copper	398		
Iron	80.4		
Nickel	89.9		
Silver	428		
Alumina	30.1	Polyethylene	0.38
Magnesia	37.7	Polypropylene	0.12
Spinel	15.0	Polystyrene	0.13
glass	1.7	Teflon	0.25
		Nylon	0.24

Assume the layer thicknesses are $L_1 = L_2 = 5$ in, $(T_{\infty,1} - T_{\infty,2}) = 80$ °C. Find q_x for the following three situations:

a) Material 1-magnesia; Material 2–magnesia;

b) Material 1-spinel; Material 2-silver;

c) Material 1-iron; Material 2-polyethylene.

7.10 The earth is cooling down due to an unforeseen disaster on the surface of the sun. The earth's engineers have undertaken a monumental task of bringing thermal energy from the center of the earth. Therefore, they have drilled deep holes and inserted copper rods $\left(k = 402 \frac{\mathrm{J}}{\mathrm{sec - m - C}}\right)$ up to the depth where the temperature is equal to the melting point of copper ($T_M = 1100$ °C), a depth of about 80 km. The plan is to have the rods insulated along their lateral boundaries.

REQUIRED:

i) Find the temperature along the copper rods, as a function of length, at the North and South Poles (-20 °C) and at the equator (30 °C) for steady state conditions.

ii) Find the heat flux (energy per unit area per unit time) for the above cases.

7.11 *GIVEN*: 1-D steady state heat flow through a slab of thickness $L = 7$ m which is *insulated on the left boundary.* Boundary temperatures are as shown on the sketch. The slab has a constant heat source of $\rho\Phi = 45\ \frac{\mathrm{W}}{\mathrm{m}^3}$.

REQUIRED: The temperature distribution $T(x)$ in the slab.

7.12 *GIVEN*: A slab as shown below.

REQUIRED: Determine $T(x)$ by integrating the ODE and applying the boundary conditions (BCs).

7.13 *GIVEN*: A slab as shown below.

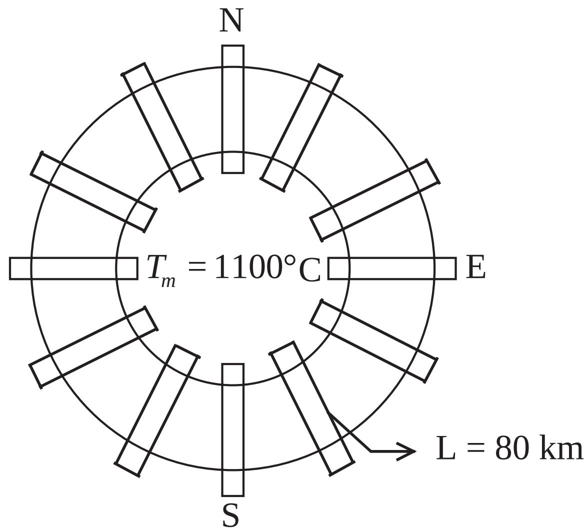
REQUIRED: Determine $T = T(x)$

7.14 A slab as shown below with convection on the left boundary and specified temperature on the right boundary.

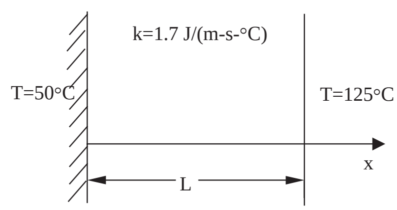
Determine $T(x)$ and the length L such that the heat flux going out the right side of the wall does not exceed 80 $\left(\frac{\mathrm{W}}{\mathrm{m}^2}\right)$.

7.15 *GIVEN*: A slab as shown below which is insulated on the left boundary and has a specified temperature on the right boundary.

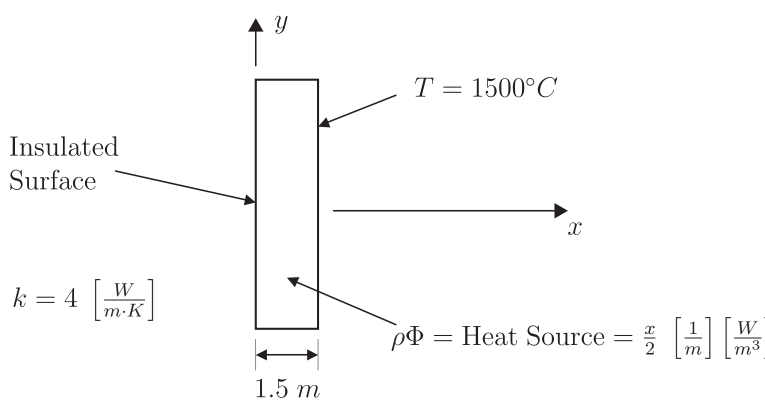
REQUIRED: Determine the temperature distribution $T(x)$ in the slab.



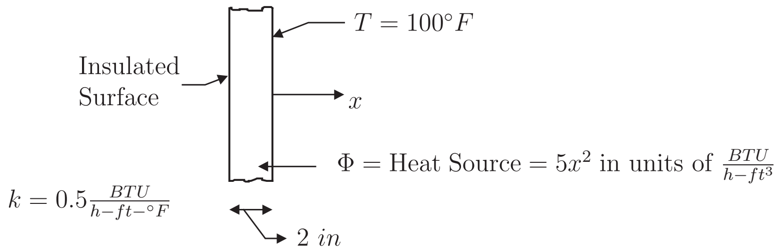
Problem 7.10



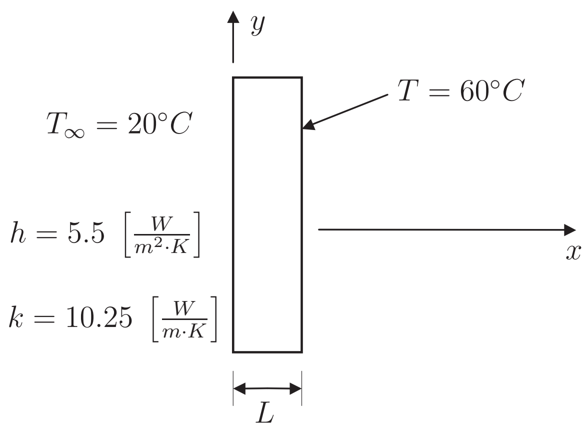
Problem 7.11



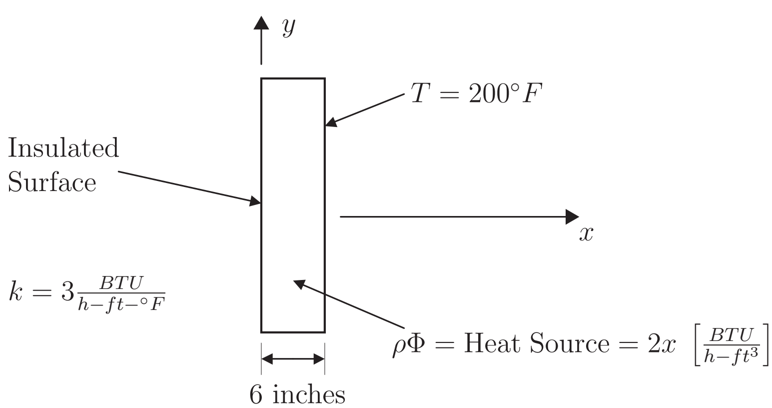
Problem 7.12



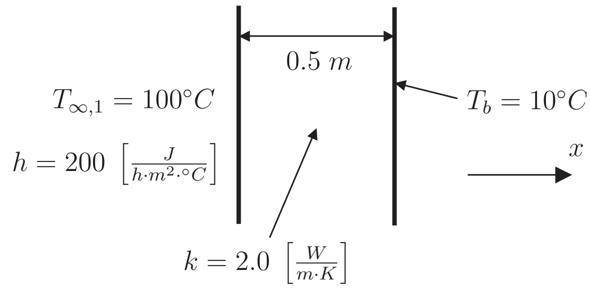
Problem 7.13



Problem 7.14



Problem 7.15



Problem 7.16

7.16 *GIVEN* The single layer slab shown below with convection on the left boundary and specified temperature on the right boundary.

REQUIRED:

a) Solve the heat transfer equation and find the temperature distribution $T(x)$ using the second-order differential equation without a heat source.

b) Write out the boundary conditions and the interface (matching) conditions required to solve the heat transfer problem.

c) Write out the boundary conditions and the interface conditions in terms of the temperature profile constants for each layer.

d) Solve for the constants using the above equations.

7.17 *GIVEN*: A slab with $k = 1.0 \ \frac{\text{W}}{(\text{m - °K})}$, and thickness $L = 1$ m. On the left surface, the temperature $T_A = 40$ °C, and on the right surface, a free convection boundary condition is applied with $h = 20 \ \frac{\text{W}}{\text{m}^2 \ \text{°K}}$ and the free stream temperature $T_\infty = 10$ °C.

REQUIRED:

a) Solve the heat transfer equation and find the temperature profile inside the slab.

b) Find the total heat loss/gain on both surfaces of the slab if it is 2.0 m high and 1 m wide.

7.18 *GIVEN*: A laterally insulated rod as shown below.

Problem 7.18

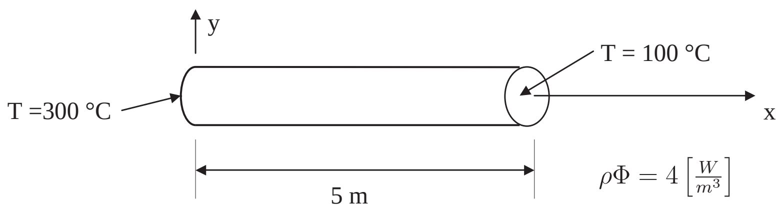
REQUIRED Determine $T = T(x)$

7.19 *GIVEN*: Same rod as in 7.18, but add an internal heat source of $\Phi = 5 \ \frac{\text{BTU}}{\text{hr - ft}^3}$.

REQUIRED:

1) Determine $T = T(x)$
2) Calculate the heat transfer rate to the surroundings.

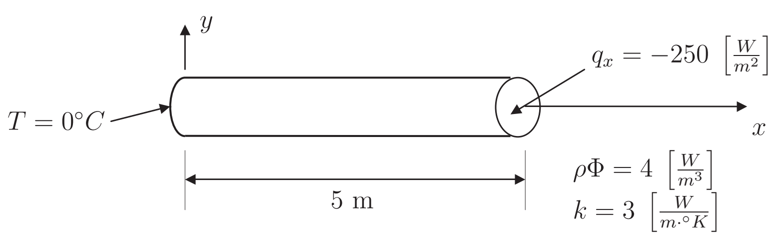
7.20 *GIVEN*: A laterally insulated cylindrical rod as shown below.



Problem 7.20

REQUIRED: Determine k such that the total heat flux at the left hand side wall does not exceed $-20\ \frac{\mathrm{W}}{\mathrm{m}^2}$. Also find the temperature profile, $T(x)$.

7.21 *GIVEN*: A laterally insulated cylindrical rod as shown below,

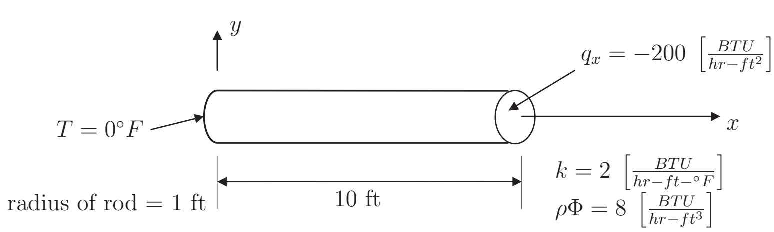


Problem 7.21

REQUIRED:

a) Determine $T(x)$
b) Determine $T(1.5\ \mathrm{m})$.

7.22 *GIVEN*: A laterally insulated cylindrical rod as shown below.



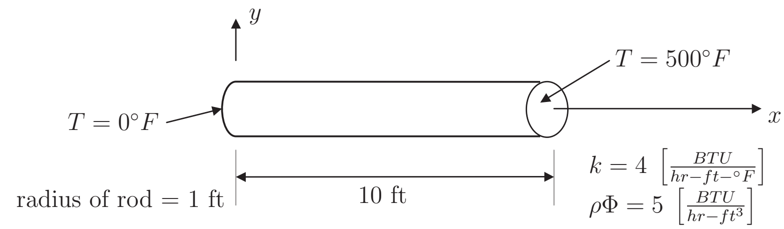
Problem 7.22

REQUIRED:

a) Determine $T(x)$

b) Determine $T(10 \text{ ft})$

7.23 *GIVEN*: A laterally insulated cylindrical rod as shown below.



Problem 7.23

REQUIRED:

a) Determine $T(x)$

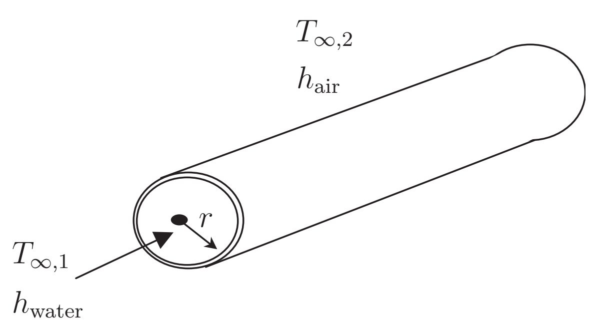
b) Calculate the heat transfer rate , Q, to the surroundings { **Note:** $Q = \int q_x dA$ and $q_x = \mathbf{n} \cdot \left(-k\frac{\partial T}{\partial x}\right)$. Also, there are two ends! }

7.24 Water at 100 °C flows through a cylindrical iron pipe of internal radius of $r_i = 5$ cm and external radius $r_o = 5.25$ cm. The air surrounding the pipe is at 25 °C. For a pipe length of 100 m. If $h_{\text{air}} = 5.0 \, \frac{\text{W}}{\text{m}^2 \, °\text{K}}$ and $h_{\text{water}} = 55.0 \, \frac{\text{W}}{\text{m}^2 \, °\text{K}}$, calculate the following:

a) Solve the ODE in order to obtain the temperature $T(r)$.

b) Calculate the heat flux $\mathbf{q}$ at r_i and r_o. Calculate at r_i and r_o the total heat loss in the pipe after one hour.

c) Calculate the temperature at the outside surface of the pipe. Is this a safe practice? From a safety point of view, what would you recommend in order to improve the design?



Problem 7.24

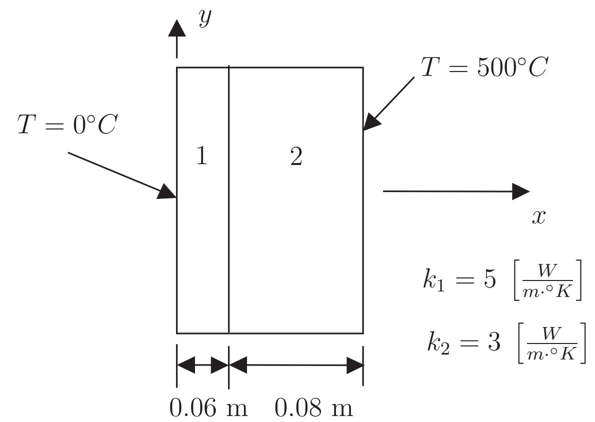
7.25 A steel pipe, with thermal conductivity of 80 $\frac{\text{W}}{\text{m} \, °\text{K}}$, has an inner diameter of 9 cm, and an outer diameter of 10 cm. The exterior of the pipe is subjected to a forced airflow at −5 °C, which produces a heat transfer coefficient of 100 $\frac{\text{W}}{\text{m}^2 \, °\text{K}}$. The tube contains flowing liquid at

50 °C and has a heat transfer coefficient of 500 $\frac{\mathrm{W}}{\mathrm{m}^2\ {}^\circ\mathrm{K}}$ at the inner pipe wall. **NOTE:** The pipe diameters are given in centimeters not meters. Determine (through integration and application of BCs) the following:

(1) the temperatures on the inner and outer surfaces of the pipe;

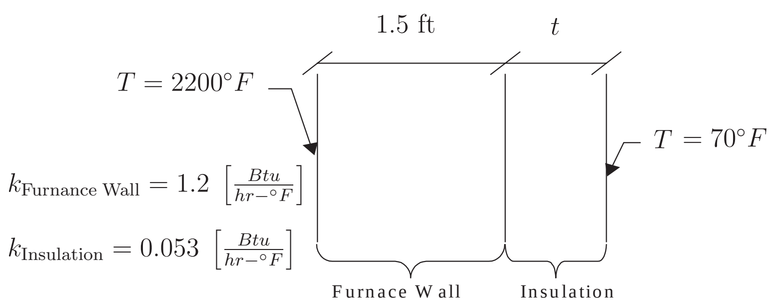
(2) the total heat loss per hour and per meter of the pipe length.

7.26 Consider the two layer slab below with specified boundary temperatures as shown. Determine $T(x)$ in each layer by solving the ODE for each slab and applying boundary conditions, i.e. solve 4 equations for 4 unknown constants of integration (c_1, c_2, c_3, and c_4).



Problem 7.26

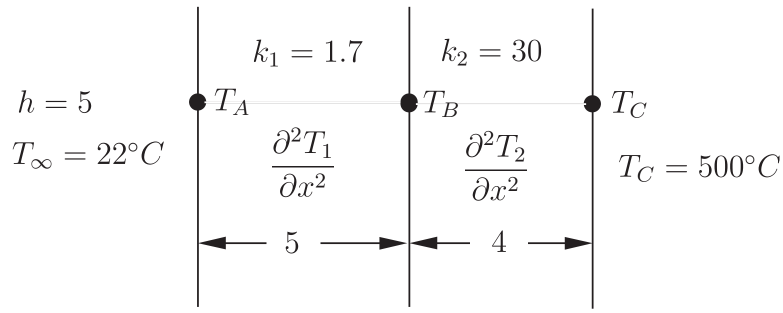
7.27 *GIVEN*: A furnace wall with specified boundary temperatures is insulated as shown:



Problem 7.27

REQUIRED: Calculate the minimum insulation thickness, t, required to maintain a heat loss of: 250 $\frac{\mathrm{BTU}}{\mathrm{hr\ -\ ft}^2}$

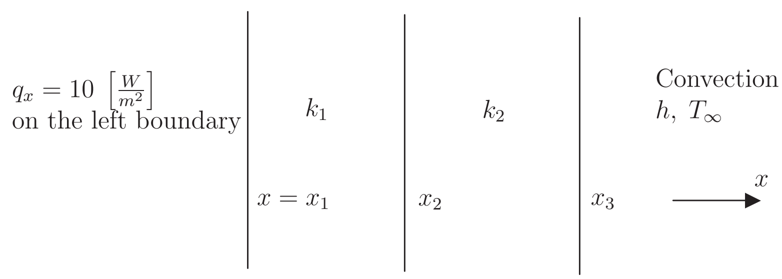
7.28 *GIVEN*: steady state conditions, 1-dimension, $\Phi = 0$, with free convection on the left boundary and specified temperature on the right boundary. Assume all quantities are metric.



Problem 7.28

FIND: Determine $T_1(x)$ and $T_2(x)$ by solving the ODE for each layer and applying the appropriate boundary conditions.

7.29 *GIVEN*: A 2-layer composite flat wall, as shown below, with given heat flux at the left surface of the wall and free convection boundary at the right surface of the wall. The material constants and the magnitude of the heat flux and the far field temperature are indicated in the figure.



Problem 7.29

REQUIRED: Write out the *boundary conditions* and the *interface conditions* required to solve the heat transfer problem (Do *not* try to solve the problem).

7.30 *GIVEN*: A 2-layer composite flat wall, as shown below, with $T_A = 22$ °C at the left surface of the wall and $T_B = 2$ °C at the right surface of the wall. The material constants are: $k_1 = 0.04 \ \frac{\mathrm{W}}{\mathrm{m\ °K}}$, $k_2 = 0.12 \ \frac{\mathrm{W}}{\mathrm{m\ °K}}$.

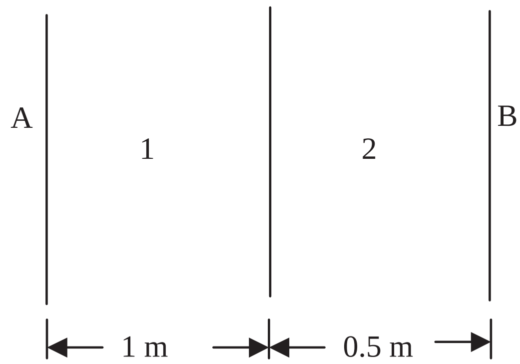
REQUIRED:

a) Derive the steady state temperature profile in each layer using the second order differential equation in the absence of any heat source.

b) Write out the *boundary conditions* and the *interface conditions* required to solve the heat transfer problem.

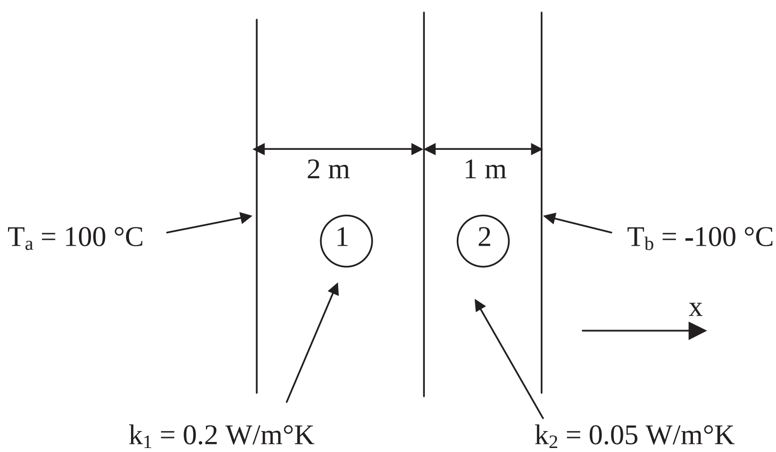
c) Write out the boundary conditions and the interface conditions in terms of the constants of temperature profile in each layer.

d) Solve for the constants using the above equations and determine the temperature profile in each layer.



Problem 7.30

7.31 *GIVEN*: The 2-layer slab shown below with temperature specified on both the left and right boundary.



Problem 7.31

REQUIRED:

a) Derive the steady state temperature profile in each layer using the second-order differential equation in the absence of any heat source.

b) Write out the boundary conditions and the interface (matching) conditions required to solve the heat transfer problem.

c) Write out the boundary conditions and the interface conditions in terms of the temperature profile constants for each layer.

d) Solve for the constants using the above equations.

7.32 A wall consists of two 1 cm thick wood board surfaces enclosing a 10 cm thick cavity filled with insulation. If the thermal conductivity of the wood board and insulation are 0.12 $\frac{\mathrm{W}}{\mathrm{m}\ ^\circ\mathrm{K}}$ and 0.04 $\frac{\mathrm{W}}{\mathrm{m}\ ^\circ\mathrm{K}}$, respectively, and free convection conditions exist on the wall exterior surfaces with a heat transfer coefficient of 2 $\frac{\mathrm{W}}{\mathrm{m}^2\ ^\circ\mathrm{K}}$. Find

a) the temperatures on the two surfaces of the wall if outside air (to right of wall) is 0 °C, and inside air (to left of wall) is 20 °C;

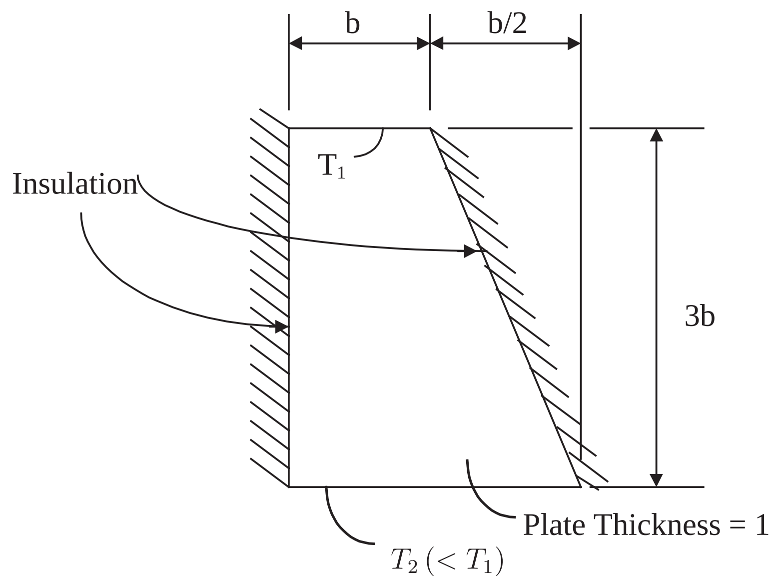
b) the effective heat transfer coefficient R for the wall;

c) the heat flux through the wall; and

d) total heat loss in an hour through the wall if the wall is 3 m high and 5 m wide.

7.33 Two-dimensional heat flow occurs in the plate shown below (heat flow is vertical). Derive the partial differential equation assuming a constant thermal conductivity k and a steady state situation. Also write out the expressions for each of the boundary conditions. Calculate the total heat imparted through the plate assuming the solution $T(x, y)$ for this problem was given (in say $\frac{\mathrm{J}}{\mathrm{hr}}$).

(Hint: At steady state, $Q_{\mathrm{in}} = Q_{\mathrm{out}}$, i.e., the amount of heat which enters the top edge (with T_1) edge over a given time period is equal to the amount which leaves along the bottom edge during the same time period.)

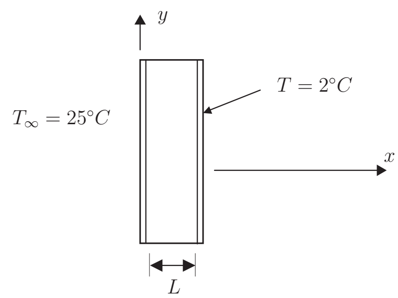


Problem 7.33

7.34 In order to design a refrigerating compartment, the following requirements are given:

* The walls will be composed by two layers of aluminum, each with a thickness of 2 cm; and a layer of insulation confined between the aluminum walls. The conductivity of the aluminum and the insulation is given by $k_{\mathrm{Al}} = 247\ \frac{\mathrm{W}}{\mathrm{m}\ ^\circ\mathrm{K}}$ and $k_{\mathrm{ins}} = 0.25\ \frac{\mathrm{W}}{\mathrm{m}\ ^\circ\mathrm{K}}$.
* The heat flux through the wall should not exceed 40 $\frac{\mathrm{W}}{\mathrm{m}^2}$.

- The temperature at the inner surface is required to be constant and equal to 2 °C, while the outside wall is exposed to convection conditions. The air outside the compartment is at 25 °C and the convection coefficient is $h = 5.5 \, \frac{\mathrm{W}}{\mathrm{m}^2 \, ^\circ\mathrm{K}}$.



Problem 7.34

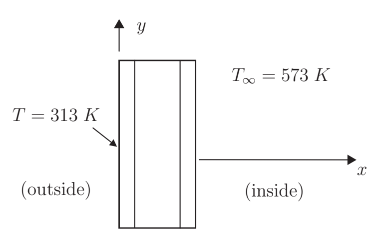
a) Determine $T(x)$ and the thickness L of the insulator such that the design satisfies all the stated requirements, AND the following:

b) Calculate the temperature on the outer surface of the wall and the heat flux through the wall.

c) Estimate the total heat loss per wall in an hour through, if the wall is 3 m high and 5 m wide.

7.35 For the oven described by the layered wall shown below, the following data is specified:

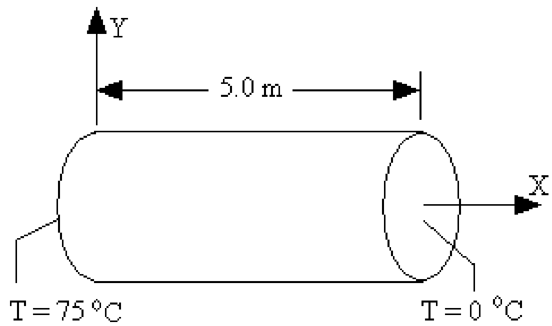
- The outer and inner layers are 2 cm each and the in between layer is 6 cm thick, for a total thickness of 10 cm.
- Thermal conductivity coefficients are given as $k_1 = 80.4 \, \frac{\mathrm{W}}{\mathrm{m} \, ^\circ\mathrm{K}}$ (outside layer), $k_2 = 1.7 \, \frac{\mathrm{W}}{\mathrm{m} \, ^\circ\mathrm{K}}$ (middle layer) and $k_3 = 80.4 \, \frac{\mathrm{W}}{\mathrm{m} \, ^\circ\mathrm{K}}$ (inside layer).
- The temperature inside the oven is 300 °C and the convection coefficient for the air inside is given as $20 \, \frac{\mathrm{W}}{\mathrm{m}^2 \, ^\circ\mathrm{K}}$. For safety reasons, the temperature of the outside *surface* should not exceed 40 °C.



Problem 7.35

REQUIRED:

Solve the corresponding ODE to obtain $T(x)$ for each layer. Estimate the heat flux and the total heat loss per wall per day, if the wall is 1 m high and 1.5 m wide.



Problem 7.36

7.36 *GIVEN*: A laterally insulated thin rod, as shown below:

A linearly increasing heat source located at the center of the rod is given by:

$$\rho\Phi = \left(x\frac{\mathrm{J}}{\mathrm{s\ m^2}} + 1\frac{\mathrm{J}}{\mathrm{s\ m^3}}\right)$$

Since the rod is insulated along it's lateral surface, heat flows only in the x direction and $T = T(x)$.

REQUIRED:

a) Assuming the rod has uniform thermal conductivity coefficient, k, determine the temperature field $T(x)$ inside the rod by integrating the governing differential equation and applying the boundary conditions.

b) Consider the two cases for thermal conductivity:

$$k_{\text{copper}} = 398\ \frac{\mathrm{J}}{\mathrm{s\ m\ {}^\circ K}} \qquad 1)$$

$$k_{\text{Nylon}} = 0.24\ \frac{\mathrm{J}}{\mathrm{s\ m\ {}^\circ K}} \qquad 2)$$

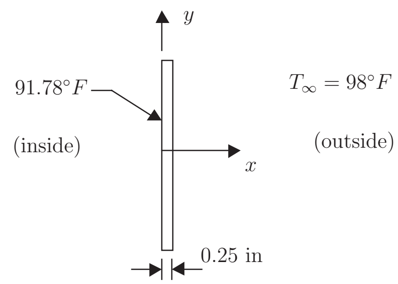
Show graphically the temperature distribution $T(x)$ vs. x for these two cases (on the same plot).

c) Calculate the heat flux in the x direction (heat transfer rate per unit area) to the surroundings at both ends.

7.37 *GIVEN*: Consider a large plate of glass in a skyscraper in downtown Houston with heat flow in the x direction only as shown below.

At the left boundary, inside an office, the temperature of the glass surface is measured to be 91.78 °F. The outside air temperature is 98 °F. The thermal conductivity for the glass is: $k = 1.7\ \frac{\mathrm{J}}{\mathrm{m\ s\ {}^\circ C}}$ and the convection coefficient for the outside air is $h = 30\ \frac{\mathrm{J}}{\mathrm{m^2\ s\ {}^\circ C}}$.

REQUIRED:



Problem 7.37

a) Integrate the governing differential equation and apply the boundary conditions to determine $T(x)$ and $q(x)$ in the glass. Note: you must convert to *consistent units* (usually easier to convert temperature and thickness to metric).

b) What is the temperature at the outside surface of the glass (in °C and °F)?

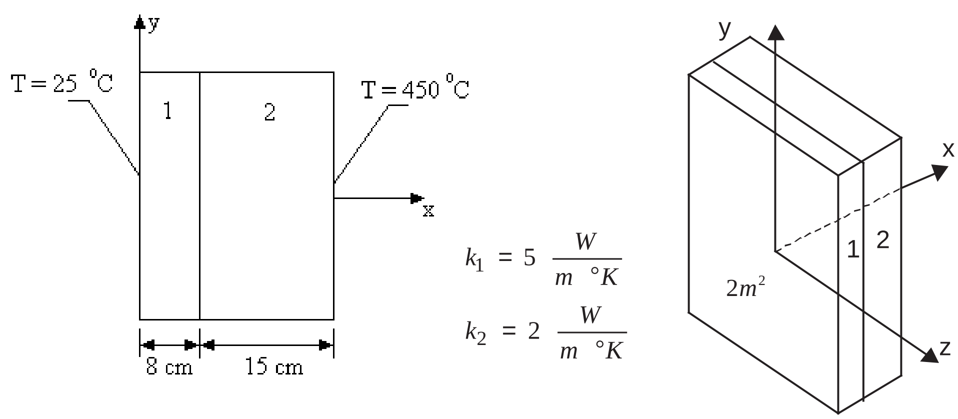
c) Plot $T(x)$ and $q(x)$ from part a.

d) How much heat energy is lost through the window in an 8 hr work day if the window is 6 ft wide by 10 ft tall.

– Note: Think about what this means as far as energy needed to keep the office at its current temperature and where that energy comes from, especially in a 70+ story building.

e) Notice that the air temperature inside the office was not needed above (because the inside glass surface temperature was known). If the convection coefficient for the inside air is $h = 10\ \frac{\mathrm{W}}{\mathrm{m^2\ °C}}$, use the results from previous parts of the problem to determine the temperature of the inside air.

7.38 *GIVEN*: Consider a two-layer slab with specified boundary temperatures as shown below. Heat is assumed to flow only in the x direction (perpendicular to the y-z plane).



Problem 7.38

REQUIRED:

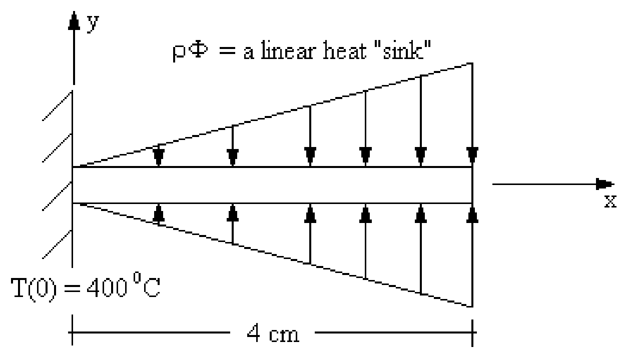
a) Determine $T(x)$ in each layer by solving the governing ordinary differential equation (ODE) for each slab and applying the boundary conditions. (i.e. solve the system of four equations and four unknowns for the four constants of integration)

b) Plot $T(x)$ and $q(x)$ for the whole system (both slabs); i.e. only one graph for $T(x)$ and one graph for $q(x)$.

c) Calculate the heat flux vector through both surfaces (left and right) and at the contact surface of the two slabs.

d) Calculate the total heat energy flow in 30 min through the right surface if the cross sectional area of the slab is 2 m^2.

7.39 *GIVEN*: The Copper cooling rod shown below:



Problem 7.39

Note: This is a 2-D problem treated as a 1-D problem!

$$\begin{aligned} k_{\text{Cu}} &= 398\ \frac{\text{J}}{\text{m s °C}} \\ \rho\Phi &= -(12500x + 250)\ \frac{\text{W}}{\text{m}^3} \qquad x \text{ in meters!!} \end{aligned}$$

REQUIRED:

1. Find and plot $T(x)$ and $q(x)$ if the heat flux at $x = 4$ cm is 25 $\frac{\text{W}}{\text{m}^2}$.
2. Find and plot $T(x)$ and $q(x)$ if the heat flux at $x = 4$ cm is 250 $\frac{\text{W}}{\text{m}^2}$.

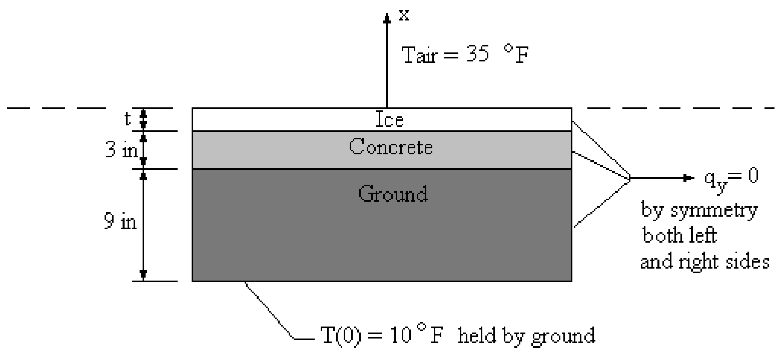
7.40 *GIVEN*: A driveway in Alberta, Canada as shown below:

$$h_{\text{air}} = 6\ \frac{\text{W}}{\text{m}^2\ °\text{K}},\ k_{\text{ground}} = 0.5\ \frac{\text{W}}{\text{m °C}},\ k_{\text{concrete}} = 1.8\ \frac{\text{W}}{\text{m °C}},\ k_{\text{ice}} = 2.2\ \frac{\text{W}}{\text{m °C}}$$

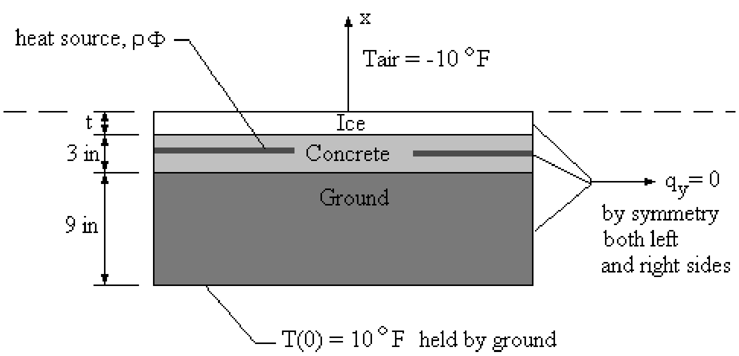
REQUIRED:

a) Calculate the thickness of the layer of ice if the heat flux due to convection is $-10\ \frac{\text{W}}{\text{m}^2}$.

b) Plot $T(x)$ for all three slabs on one graph and denote (with a horizontal line if x is the vertical axis) the locations where each slab starts and ends.



Problem 7.40



Problem 7.41

c) The same as part b) but for $q(x)$.

7.41 *GIVEN*: A heated driveway in Alberta, Canada as shown below:

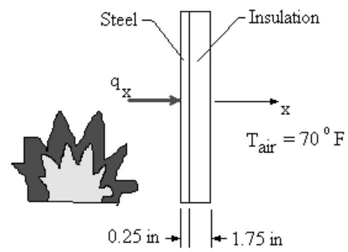
$$h_{\text{air}} = 6 \, \frac{\text{W}}{\text{m}^2 \, {}^\circ\text{K}}, \; k_{\text{ground}} = 0.5 \, \frac{\text{W}}{\text{m} \, {}^\circ\text{C}}, \; k_{\text{concrete}} = 1.8 \, \frac{\text{W}}{\text{m} \, {}^\circ\text{C}}, \; k_{\text{ice}} = 2.2 \, \frac{\text{W}}{\text{m} \, {}^\circ\text{C}}$$

REQUIRED:

a) If $t = 1$ in, what is the minimum $\rho\Phi$ $\left(\frac{\text{W}}{\text{m}^3}\right)$ to just melt the ice?
b) Plot $T(x)$ for all three slabs on one graph and denote (with a horizontal line if x is the vertical axis) the locations where each slab starts and ends.
c) The same as part b) but for $q(x)$.

NOTE: Heat source should be considered to apply uniformly in the x direction throughout the concrete.

7.42 *GIVEN*: A two layered fire door comprised of a thin layer of steel and an insulator as shown below:



Problem 7.42

The heat flux on the left is $q_x = 918.5\ \frac{\mathrm{W}}{\mathrm{m}^2}$. PAY CLOSE ATTENTION TO UNITS.

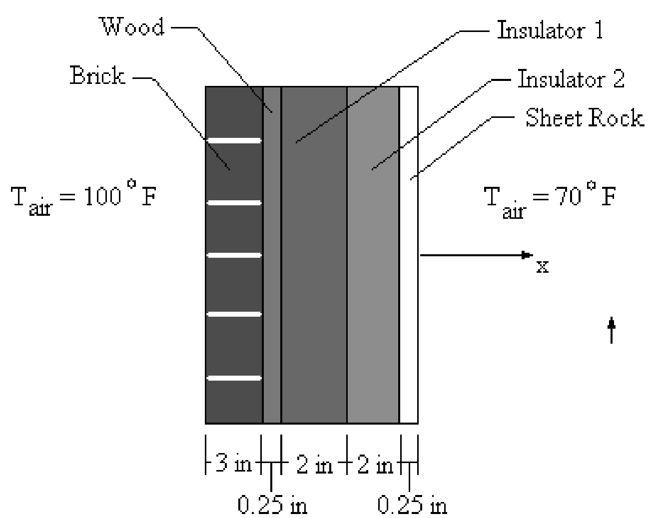
$$h_{\mathrm{air}} = 5.5\ \frac{\mathrm{W}}{\mathrm{m}^2\ {}^\circ\mathrm{K}},\ k_{\mathrm{steel}} = 46.73\ \frac{\mathrm{W}}{\mathrm{m}\ {}^\circ\mathrm{C}},\ k_{\mathrm{insul}} = 0.04\ \frac{\mathrm{W}}{\mathrm{m}\ {}^\circ\mathrm{K}}$$

REQUIRED:

a) Solve the governing ODEs for each layer and determine:
   Determine $T(x)$ for each layer and plot $T(x)$ vs x.
   Determine $q(x)$ for each layer.
   Determine the heat energy flow through $3' \times 7'$ door in 1 hour.

b) Same as part a) except solve by the effective resistance method.

7.43 *GIVEN*: The multi-layer cross section of a house wall (5-6) comprised of brick, wood, insulation, and sheet rock as shown below:



Problem 7.43

PAY CLOSE ATTENTION TO UNITS.

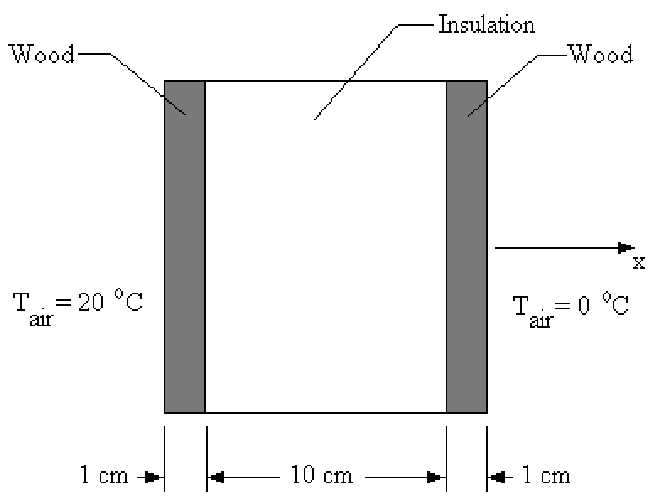
$$h_{\text{air}} = 5.5\ \frac{\text{W}}{\text{m}^2\ {}^\circ\text{K}},\ k_{\text{brick}} = 0.0066\ \frac{\text{W}}{\text{cm}\ {}^\circ\text{K}},\ k_{\text{wood}} = 0.0010\ \frac{\text{W}}{\text{cm}\ {}^\circ\text{K}},$$

$$k_{\text{ins1}} = 0.25\ \frac{\text{W}}{\text{m}\ {}^\circ\text{K}},\ k_{\text{ins2}} = 0.04\ \frac{\text{W}}{\text{m}\ {}^\circ\text{K}},\ k_{\text{sheatrock}} = 1.1\ \frac{\text{W}}{\text{m}\ {}^\circ\text{C}},$$

REQUIRED:

a) Determine effective resistance, R, for the composite wall.
b) Determine q for the composite wall.
c) Determine the temperature at each boundary and interface.
d) Plot $T(x)$ vs. x for the composite wall.

7.44 *GIVEN*: A wall consists of two 1 cm thick wood board surfaces enclosing a 10 cm thick cavity filled with insulation as shown below:



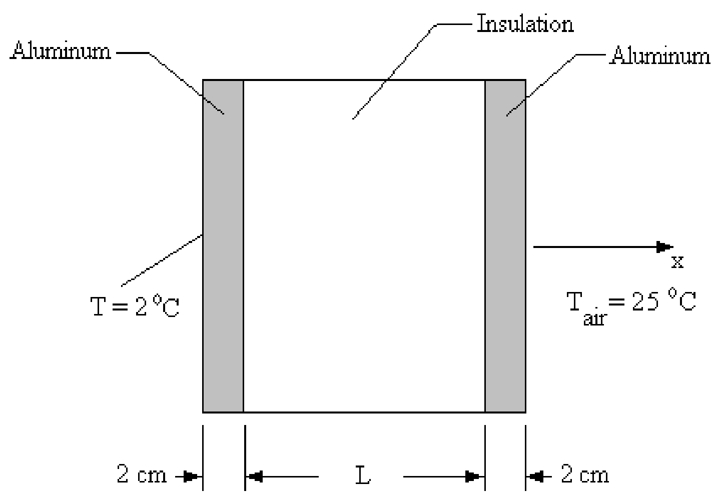
Problem 7.44

REQUIRED: If the thermal conductivity of the wood board and insulation are 0.12 $\frac{\text{W}}{\text{m}\ {}^\circ\text{K}}$ and 0.04 $\frac{\text{W}}{\text{m}\ {}^\circ\text{K}}$, respectively, and free convection conditions exist on the wall exterior surfaces with a heat transfer coefficient of 2 $\frac{\text{W}}{\text{m}^2\ {}^\circ\text{K}}$, find:

a) The temperatures on the two surfaces of the wall if the outside air (to the right of the wall) is 0 °C, and inside air (to the left of the wall) is 20 °C.
b) The effective heat transfer coefficient, R, for the wall.
c) The heat flux through the wall.
d) The total heat loss in an hour through the wall if the wall is 3 m high and 5 m wide.

7.45 *GIVEN*: In order to design a refrigerating compartment, the following requirements are given:

1. The walls will be composed of two layers of aluminum, each with a thickness of 2 cm; and a layer of insulation confined between the aluminum walls. The conductivity of the aluminum and the insulation is given by $k_{\text{al}} = 247 \, \frac{\text{W}}{\text{m °K}}$ and $k_{\text{ins}} = 0.25 \, \frac{\text{W}}{\text{m °K}}$.
2. The heat flux through the wall should not exceed $40 \, \frac{\text{W}}{\text{m}^2}$.
3. The temperature at the inner surface is required to be constant and equal to 2 °C, while the outside wall is exposed to convection conditions with an outside air temperature of 25 °C and the convection coefficient is $h = 5.5 \, \frac{\text{W}}{\text{m}^2 \text{ °K}}$.



Problem 7.45

REQUIRED:

a) Determine $T(x)$ and the thickness L of the insulator such that the design satisfies all the stated requirements.

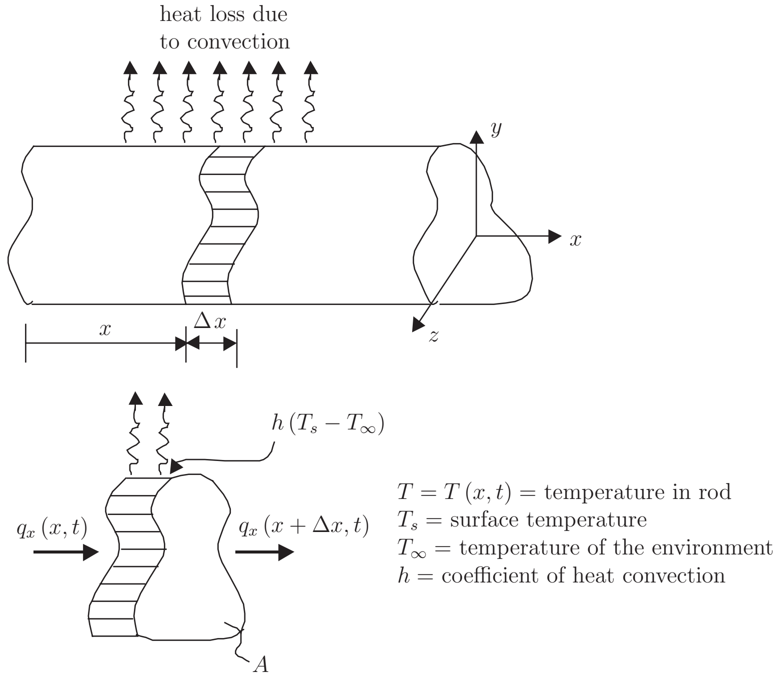
b) Calculate the temperature of the outer surface of the wall and the heat flux through the wall.

c) Calculate the total heat loss per wall in an hour through, if the wall is 3 m high and 5 m wide.

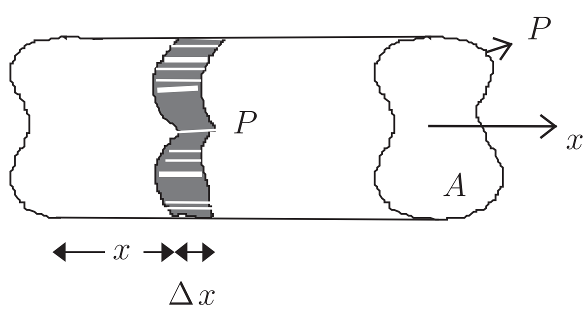
7.46 Consider the case of heat transfer in a thin un-insultated rod shown below. We wish to determine a simplified solution for this problem.

For this 3-D problem, the heat flux in each coordinate direction is given by:

$$\left.\begin{aligned} q_x &= -k\tfrac{\partial T}{\partial x} \\ q_y &= -k\tfrac{\partial T}{\partial y}\Big|_{y=\pm\frac{a}{2}} = h\left(T\left(x,\pm\tfrac{a}{2},z\right) - T_\infty\right) \\ q_z &= -k\tfrac{\partial T}{\partial z}\Big|_{z=\pm\frac{a}{2}} = h\left(T\left(x,y,\pm\tfrac{a}{2}\right) - T_\infty\right) \end{aligned}\right| \quad \begin{aligned} T(0,y,z) &= T_0 \\ T(L,y,z) &= T_L \end{aligned}$$



Problem 7.46



Problem 7.40a

While the solution for $T(\,x, y, z\,)$ can be attempted by solving the heat transfer equation in 3-D, this is difficult because of the mathematics. However, we can simplify the problem by making the observation that most of the heat flow will be in the direction of the solid due to conduction (in the x direction). Only a small amount of heat flow will occur normal to the x-axis within the solid by conduction and out the perimeter of the solid rod by convection. Consequently, to approximately solve the heat conduction equation in an un-insulated rod, one assumes that it is a one-dimensional solid and that the temperature distribution is approximately 1-D, i.e., $T = T(x)$ only for the steady state and $T(\,x, t\,)$ for the time-dependent case. The heat conduction equation becomes:

$$\rho\hat{C}\frac{\partial T}{\partial t} = +k\nabla^2 T + \rho\Phi \implies \rho\hat{C}\frac{\partial T}{\partial t}(\,x,t\,) = k\frac{\partial^2 T}{\partial x^2}(\,x,t\,) + \rho\Phi(\,x,t\,)$$

However, since the perimeter is un-insulated, there will be heat *loss* through the perimeter boundary that must be account for. We can do this by defining a heat loss term Φ for the 1-D geometry whose magnitude is equal to the heat loss that would occur in the actual 3-D problem. The term $\rho\Phi$ can be approximately taken to be equal to

$$\rho\Phi = \frac{-h(T - T_\infty)P\Delta x}{A\Delta x} = -h(T - T_\infty)\frac{P}{A} \qquad \left(\frac{\mathrm{J}}{\mathrm{m}^3\text{ - s}}\right)$$

where P is the perimeter of the boundary as shown above.

a) Explain and justify the use of the heat loss term.

b) Determine the solution for $T(x)$ for the steady state case.