From Conservation to Kirchhoff: Getting Started in Circuits With Conservation and Accounting

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Abstract

This paper explores the connection between the general "conservation and accounting" approach to engineering science that forms the primary theme of the Foundation Coalition Sophomore Engineering Curriculum at Rose-Hulman Institute of Technology and the more specialized "Kirchhoff's laws" techniques that are basic for understanding electric circuits.

Circuit theory exists as a distinct discipline because of the set of specialized problem-solving techniques that apply to electric circuits. Conservation and accounting techniques provide a powerful set of tools for engineering problem solving, but conservation of charge and conservation of energy are not as useful as Kirchhoff's laws for routine solution of electrical circuit problems. Kirchhoff's laws are not wholly a consequence of the conservation principles. The laws also depend on the "lumped circuit" assumptions that restrict the class of problems to which circuit theory applies.

The paper introduces the circuit variables current and voltage, and examines their relation to the conserved quantities of charge and energy. The lumped circuit model is discussed to illustrate the circumstances under which Kirchhoff's laws apply.

Introduction

This paper explores the connection between the "conservation and accounting" approach to engineering science that forms the primary theme of the Foundation Coalition Sophomore Engineering Curriculum at Rose-Hulman Institute of Technology [1] and the more specialized "Kirchhoff's laws" techniques that are basic for understanding electric circuits. The Foundation Coalition Sophomore Engineering Curriculum at Rose-Hulman consists of a sequence of eight coordinated courses, including five in engineering science. These courses consist of Conservation and Accounting Principles in the Fall Quarter, followed by Electrical Systems, Thermal and Fluid Systems, and Mechanical Systems running in parallel in the Winter, and finishing with Analysis and Design of Engineering Systems in the Spring. This sequence of courses is required of all students majoring in electrical and computer engineering, and is elective for students majoring in mechanical, civil, and chemical engineering. Conservation of mass, charge, linear and angular momentum, and energy, and accounting of species mass and entropy are the fundamental principles that provide a unified introduction to engineering problem-solving [2]. The three winter term "systems" courses, however, offer an opportunity to introduce techniques that are specific to distinct classes of problems.

Circuit theory exists as a distinct engineering discipline because of the set of specialized problem-solving techniques that apply to electric circuits. Conservation and accounting techniques provide a powerful and very general set of tools for engineering problem solving, but conservation of charge and energy are not as effective as Kirchhoff's laws for everyday solution of electrical circuit problems. An analogous situation exists in other branches of engineering. In mechanics, kinematics complements conservation of energy and momentum for solving dynamics problems.

It is tempting to suppose that Kirchhoff's laws are simply a restatement of conservation of charge and energy, but this is not the case. The laws depend in fact on the "lumped circuit" assumptions that restrict the class of problems to which circuit theory applies. Although Kirchhoff's current law is associated with the conservation of charge, it requires the additional assumption that charge does not build up on a circuit node. Kirchhoff's voltage law rests on an assumption that time-varying magnetic flux does not link the circuit loops. These laws are useful in modeling physical circuits because of the way that circuits are built. It is entirely possible to build electrical circuits that violate Kirchhoff's laws. In fact "distributed" circuit models are routinely used in microwave applications (high frequencies) and power transmission (large size). The principal advantage to "lumped" circuits is that a considerable body of analytical technique has been developed over the years for dealing with them.

This document is divided into two parts. The first part shows how the circuit variables current and voltage can be introduced and related to the conserved quantities of charge and energy. The second part of the document discusses the lumped circuit model, and describes the circumstances under which Kirchhoff's laws apply.

The Circuit Variables

Charge and Current

Consider a "system" or control volume as shown in Fig. 1. Wires are shown so that charge can be moved into or out of the system. The *accounting statement for charge* says that

$$\frac{dq_{sys}}{dt} = \dot{q}_{in} - \dot{q}_{out} + \dot{q}_{gen} - \dot{q}_{cons}\ , \qquad (1)$$

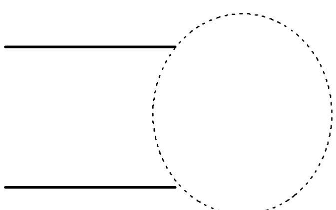

Figure 1. A Control Volume for Charge

where q_{sys} is the net charge accumulated within the system, $\dot{q}_{in}$ is the rate of flow of net charge into the system from all sources, $\dot{q}_{out}$ is the rate of flow of net charge out of the system, $\dot{q}_{gen}$ is the rate at which charge is generated within the system, and $\dot{q}_{cons}$ is the rate at which charge is consumed within the system. Now any positive charge that is generated within the system must be either consumed within the system or matched by the generation of negative charge at an equal rate. A similar statement applies to negative charge generated within the system. This is the statement of *conservation of charge*. Analytically, conservation of charge requires that the last two terms of equation (1) sum to zero. The accounting statement for charge becomes

$$\frac{dq_{sys}}{dt} = \dot{q}_{in} - \dot{q}_{out} = \dot{q}_{in,net} \qquad (2)$$

Now current is the rate of flow of charge. That is

$$i_{in,net} = \dot{q}_{in,net}\ . \qquad (3)$$

Substituting in equation (2) gives

$$\frac{dq_{sys}}{dt} = i_{in,net}\ . \qquad (4)$$

This is the correct form of the equation that is usually presented as *"i = dq/dt"* in circuit theory textbooks.

Since analysis of a circuit may involve consideration of a number of "systems" of circuit components, it is not always clear which direction of current flow is "in." Normally a reference direction for current flow is designated with an arrow, as in Fig. 2. Positive current is understood to be current flowing in the direction of the arrow.

While it is possible physically for charge to build up inside a device, unlimited charge buildup can develop into a safety hazard. (What happens when charge builds up in a cloud?) Consequently, most electrical devices are designed so that $\frac{dq_{sys}}{dt} = 0$. A

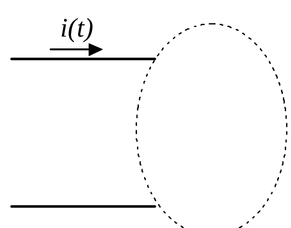

Figure 2. Reference Direction for Current

typical case is depicted in Fig. 2, where if a current *i(t)* flows in through one of the wires, an equal current flows out the other. A pair of wires carrying equal and opposite currents that allows access to a device is called a *port.*

Voltage, Energy, and Power

A charge placed in an electric field will be acted on by a force. The magnitude of the force is proportional to the amount of the charge. Because of this force, work is required to move the charge over a distance in the electric field. We can define the *voltage* between two points, A and B, as the work per unit charge required to move a charge from B to A. If the charge is positive and an external agent does work on the charge in moving it from B to A, then we say that A is positive with respect to B. Voltage is more formally known as *electric potential*. We designate the electric potential of A with respect to B as v_{AB} .

A charge q in an electric field experiences a change in electrical potential energy of $DE = q(v_{AB})$ when moved from point B to point A. (Do not confuse the *potential energy* $q(v_{AB})$ with the *potential* v_{AB} .) There is a close analogy with raising a mass in a gravitational field. Just as gravitational potential energy can be converted to other forms of energy or to work by allowing the gravitational field to act in moving the mass from A back to B, the electrical potential energy can be converted to other forms of energy or to work by allowing the electric field to act in moving the charge q from A back to B.

Electrical potential energy, like all forms of energy, is a relative quantity and must be specified with respect to a reference. This means that electric potential must be specified with respect to a reference as well. In circuit analysis problems it is often convenient to designate a single point in the circuit as "ground" or "neutral," and to reference all potentials to this point. The potential of point A with respect to neutral is written v_A , or v_{AN} . We then have $v_{AB} = v_{AN} - v_{BN}$.

Now consider the circuit shown in Fig. 3, in which a single element or group of elements has been identified as the "system." The potential $v(t)$ shown in the diagram might also have been written as $v_{AB}(t)$. The potential $v(t) = v_{AN}(t) - v_{BN}(t)$ is shown with a *reference polarity*. A positive value of $v(t)$ implies that the potential of A with respect to B is positive.

Now suppose that the wires connected to the "system" constitute a port, so that equal and opposite currents flow and no charge accumulates in the system. A charge q leaving the system at point B carries out an electrical potential energy of qv_{BN} . A charge q entering the system at point A carries in an electrical potential energy of qv_{AN} . If charge flows out of the system at point B at the rate $i = \dot{q}$, then in a small time dt an amount of charge equal to $\dot{q}\,dt$ will have left the system. This charge carries out with it an amount of potential energy equal to $\dot{q}v_{BN}\,dt$. At point A charge is flowing into the system at the rate $i = \dot{q}$. In a small time dt an amount of charge equal to $\dot{q}\,dt$ enters the system here. This charge carries in with it an amount of potential energy equal to

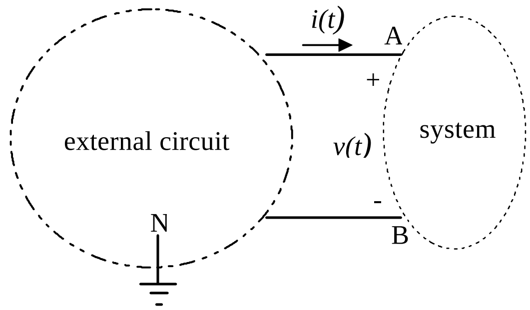

Figure 3. Voltage and Current at a Port

$\dot{q}v_{AN}\,dt$. The net increase in electrical potential energy in the system in time dt is

$$dE = \dot{q}v_{AN}\,dt - \dot{q}v_{BN}\,dt = \dot{q}v\,dt = iv\,dt$$

We now have the rate equation

$$\frac{dE}{dt} = iv\,, \qquad (5)$$

where the right hand side represents the rate at which electrical energy flows into the system through the port AB. Since the rate of flow of energy is *power*, we can write

$$\frac{dE}{dt} = i(t)v(t) = p(t). \qquad (6)$$

Equation (6) is only a part of the accounting equation for electrical energy, as the energy that flows into the system through port AB may not all be accumulated inside. More completely we have

$$\frac{dE}{dt} = \sum p(t) + \sum \dot{W}_{gen} - \sum \dot{F}_{cons}\,, \qquad (7)$$

where $p(t)$ represents the flow of electrical power into the system through a port, $\dot{W}_{gen}$ represents the rate at which electrical energy is generated within the system (e.g. by conversion from mechanical or chemical energy) and $\dot{F}_{cons}$ represents the rate at which electrical energy is consumed within the system (by conversion to mechanical or chemical energy, or to heat). A term for electromagnetic radiation can be added. The parameter E represents energy stored within the system, either as electrical energy or as magnetic energy.

The Lumped Circuit Model

Classical circuit theory incorporates techniques for analyzing circuits consisting of discrete components ("lumps") interconnected by wires. The most important attribute of the lumped circuit model is that Kirchhoff's two celebrated circuit laws apply to these circuits.

The central assumption of the lumped circuit model is that electrical and magnetic energy are stored or converted to other forms of energy only within the circuit elements. If we associate energy storage with the electromagnetic fields, our assumption is that no electric fields exit in the space outside the elements and that no time-varying magnetic flux intersects any of the circuit loops. We also assume that a lumped circuit is physically "small," in a sense to be described.

Suppose we imagine a "gaussian surface" enclosing any collection of circuit elements. If there were ever a net accumulation of charge within such a surface, there would be an electric field in the space surrounding the surface. This would violate our assumption that such fields are confined within the devices. Thus in a lumped circuit there can be no such accumulation of charge.

An important special case is when the gaussian surface encloses a circuit node, as shown in Fig. 4. Equation (4) tells us that

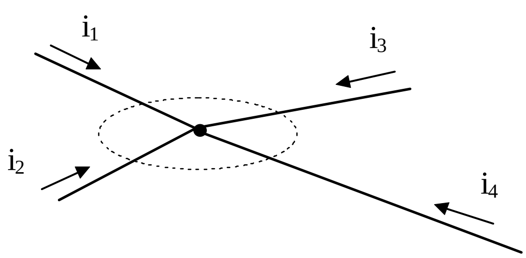

Figure 4. A Circuit Node

$$\frac{dq_{sys}}{dt} = i_1 + i_2 + i_3 + i_4 \,, \qquad (8)$$

where the right hand side includes all currents flowing into the node in question. Since there can be no accumulation of charge within the surface that encloses this node, the left hand side of the equation is zero. The general statement is Kirchhoff's Current Law: *The sum of the currents entering a circuit node is zero*

The second assumption of the lumped circuit model is that no time-varying magnetic flux cuts any of the circuit loops. The direct consequence of this assumption is Kirchhoff's Voltage Law: *The sum of the voltage drops around any circuit loop is zero*. No conservation law prevents time-varying magnetic flux from passing through loops of a circuit. Should this be the case, however, the lumped circuit model will provide a poor prediction of the circuit behavior.

The assumption that no time-varying magnetic flux cuts any of the circuit loops is problematic, since currents flowing in the circuit wires of necessity generate magnetic fields that intersect the circuit loops! To allow us to ignore the effects of these "self-generated" fields, we must assume either that the area of each circuit loop is small enough so that the net flux passing through the loop is small, or that the time rate of change of the flux is small. If the currents that flow in the circuit are changing with time only slowly, then we can relax the restriction on the circuit size. Conversely, if the circuit is carrying "high-frequency" currents, then it must be physically small to remain a "lumped circuit."

Conclusion

This paper has examined the relation between "conservation and accounting" and classical circuit theory by introducing the variables of circuit theory and exploring what conservation of charge and energy can tell us about their behavior. We have also identified additional assumptions that are appropriate so that Kirchhoff's laws can replace the conservation laws as the primary analysis tools. We should probably have finished up with Tellegen's theorem, the most general circuit-theoretic statement of conservation of energy in terms of voltage and current, but as this celebrated theorem is beyond the level of the Electrical Systems course, the interested reader is referred to the literature [3].

References

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