Chapter 1

INTRODUCTION

Figure 1.1:

1.1 Motivation for a Closer View

In ENGR 211 (Conservation Principles In Engineering Mechanics), conservation laws were developed from a common accounting framework statement:

$$\left(\begin{array}{l}\text{Accumulation of E.P.}\\ \text{within system}\\ \text{during time period}\end{array}\right) = \left(\begin{array}{l}\text{Amount of E.P.}\\ \text{entering system}\\ \text{during time period}\end{array}\right) - \left(\begin{array}{l}\text{Amount of E.P.}\\ \text{leaving system}\\ \text{during time period}\end{array}\right) + \left(\begin{array}{l}\text{Amount of E.P.}\\ \text{generated within system}\\ \text{during time period}\end{array}\right) - \left(\begin{array}{l}\text{Amount of E.P.}\\ \text{consumed within system}\\ \text{during time period}\end{array}\right) \tag{1.1}$$

where E.P. stands for "extensive property", like mass, linear momentum, angular momentum, or energy, and where

$$\text{Accumulation} = \left(\begin{array}{l}\text{amount at end}\\ \text{of time period}\end{array}\right) - \left(\begin{array}{l}\text{amount at beginning}\\ \text{of time period}\end{array}\right). \tag{1.2}$$

It is extremely important to note that in the general statement of the conservation laws above, the term *system* can refer to any region of space one defines it to be. In ENGR 211 the system was chosen to be of macroscopic size; e.g., a length of pipe, a tank, a vehicle, a bridge, a block sliding on an inclined plane, a pulley system, a translating and/or rotating rigid body, etc. As will be discussed soon, it may be advantageous and also necessary to define a system of microscopic size and thereby obtain a better understanding of the nature of the extensive property and other variables within the system.

The above accounting framework led to four fundamental conservation laws:

- Conservation of Mass (COM)
- Conservation of Linear Momentum (COLM)
- Conservation of Angular Momentum (COAM)
- Conservation of Energy (COE)

For example, Conservation of Mass was written in word form as (assuming no generation or consumption terms):

$$\left(\begin{array}{l}\text{Mass accumulation}\\ \text{within system}\\ \text{during time period}\end{array}\right) = \left(\begin{array}{l}\text{Mass entering}\\ \text{system during}\\ \text{time period}\end{array}\right) - \left(\begin{array}{l}\text{Mass leaving}\\ \text{system during}\\ \text{time period}\end{array}\right), \tag{1.3}$$

where

$$\left(\begin{array}{l}\text{Mass accumulation}\\ \text{within system}\\ \text{during time period}\end{array}\right) = \left(\begin{array}{l}\text{Mass contained}\\ \text{in system at}\\ \text{end of time period}\end{array}\right) - \left(\begin{array}{l}\text{Mass contained}\\ \text{in system at}\\ \text{beginning of}\\ \text{time period}\end{array}\right). \tag{1.4}$$

In mathematical form, Conservation of Mass may be written as

$$(m_{sys})_{end} - (m_{sys})_{beg} = \sum m_{in} - \sum m_{out} \tag{1.5}$$

and in rate form as

$$\frac{dm_{sys}}{dt} = \sum \dot{m}_{in} - \sum \dot{m}_{out}, \tag{1.6}$$

where the dot on top of a symbol designates the time derivative.

The Conservation of Linear Momentum was written in rate form as

$$\frac{d\mathbf{P}_{sys}}{dt} = \sum(\dot{m}\mathbf{v})_{in} - \sum(\dot{m}\mathbf{v})_{out} + \sum \mathbf{F}_{ext}, \tag{1.7}$$

where

$$\begin{aligned} \mathbf{P}_{sys} &= \text{linear momentum of the system} = m\mathbf{v} \\ \sum \mathbf{F}_{ext} &= \text{external forces on system,} \end{aligned}$$

and the Conservation of Angular Momentum was written in rate form as

$$\frac{d\mathbf{L}_{sys}}{dt} = \sum(\mathbf{r} \times \dot{m}\mathbf{v})_{in} - \sum(\mathbf{r} \times \dot{m}\mathbf{v})_{out} + \sum \mathbf{T}_{ext}, \tag{1.8}$$

where

$$\begin{aligned} \mathbf{L}_{sys} &= \text{angular momentum of the system} = \mathbf{r} \times m\mathbf{v} \\ \sum \mathbf{T}_{ext} &= \text{external torques on system boundary.} \end{aligned}$$

It should be recalled that equations (1.7) and (1.8) are general vector equations valid in any coordinate system. In ENGR 211, we obtained special forms of the conservation equations for translation and rotation of a rigid body with respect to the center of mass of the body (as well as with respect to other reference points).

In developing and applying these conservation principles in ENGR 211, attention was restricted to *macroscopic* systems. Some of these were:

- Static fluid in a tank and fluid pressure on a submerged body
- Fluid flow in/out of a tank or pipe
- Static analysis of rigid bodies such as blocks, truss members, and frame members
- Dynamic analysis of translating and rotating bodies and systems which included lumped masses, rigid bars, rigid disks and other bodies.

In considering problems such as those listed above, we took an extremely important step in the specification of the system to be considered. We chose the system large enough so that we could effectively ignore the spatial variation of variables such as mass density, internal forces, velocity, energy, etc. within the system and, in some cases, also on the system boundary. For example, when we considered the analysis of a truss structure, we assumed that each truss member carried an internal tensile or compressive force over the member's cross-sectional area. However, we never considered the possibility that this force may be distributed over the cross-sectional area in some fashion. For frame structures, we assumed that reactions at supports were axial and shear forces plus moments. In considering fluid flow through a pipe or tank, we never considered what actually happened within the pipe or tank, we only considered the net accumulation of mass, momentum or energy within the system *but not their distribution within the system.* Similarly, we only considered the net amount of mass, momentum or energy crossing the system boundary, *but not their distribution over the system boundary.* Also, we only considered point external forces or torques acting on the system boundary *but not a distributed external force or torque system* (recall that distributed forces were replaced by equivalent forces and moments).

Such assumptions on and approximations to the distribution of variables within the system and on the system boundary may at first lead to the question "Was the macroscopic view used in ENGR 211 accurate enough?" The answer is "It depends." In many cases, a global or "big- picture" view of a problem leads to perfectly acceptable engineering results that allow one to characterize the overall behavior of the system. However, situations exist where a more detailed view of the problem is necessary. Consider the example of steady-state fluid flow through a straight, circular pipe. In ENGR 211, we simply considered the fluid flow out of the pipe to have a certain volumetric flow rate. Thus we assumed an average fluid velocity at the pipe exit. However, experimental measurements or experimental flow visualization will show that most fluids are viscous and produce frictional forces within the fluid and on boundary surfaces over which the flow passes. A viscous fluid (such as water or oil) will want to stick to the boundary over which it flows. In fact, even air exhibits a measurable amount of viscous behavior. Hence, the fluid velocity would be zero on the boundary of the pipe and have a maximum velocity at the center of the circular pipe. Notice that we have now postulated that the velocity will vary over the cross-sectional area of the pipe. In the most general case, the fluid velocity will vary with position along the length of the pipe as well as over the cross-sectional area (i.e., with respect to all three coordinate directions) and with time. In addition, we may want (or need) to account for fluid compressibility (i.e., non-constant density).

In ENGR 211, the effects of fluid viscosity were neglected and the assumption of an average fluid velocity over a cross-section was "acceptable". However, neglecting viscous effects also meant that we ignored energy losses due to heat generation (friction) and momentum transfer from the fluid to the system boundary (due to frictional forces on the boundary). Neglecting viscous effects would also mean that the fluid velocity and pressure would be constant along the length of the pipe, which, from experimental observations, is not the case for the majority of fluids. Consequently, in many applications, such an approximate macroscopic analysis is unacceptable.

It is resonable to ask, "What must be done in order to more precisely characterize the spatial and temporal (time) variation of variables within a system." The answer is that we must define and analyze a sub-system that is much smaller in size than the characteristic length scale of the whole system, and we must determine equations that define the spatial and temporal variation of the variables of interest. As we shall see, this means that the system must be defined as a differential volume (in the calculus sense), and differential equations must be obtained which characterize the change in mass, linear and angular momentum, and energy from point to point and with respect to time. As we shall also see, we will need to obtain certain relations between variables (called constitutive relations) that define the input and output responses of each material (e.g., viscosity coefficient, thermal conductivity, and elastic modulus).

1.2 Macroscopic vs. Microscopic

In this text we will focus on the construction and application of the conservation principles for a *continuum*. As we will see in more detail in Chapter 2, a continuum is a material system that can be divided into smaller subsystems of arbitrary small size for which all intensive properties (e.g., mass density) and other variables (e.g., velocity) are functions of position and time. Consequently, we must define the conservation laws for a point in space instead of defining it for a finite system. In other words, we are interested in defining conservation laws for the microscopic system as opposed to the macroscopic system. We will discuss some examples from fluids and solids to demonstrate the approach required in a continuum.

1.2.1 Fluids

It is instructive to explain the difference between the macroscopic and the microscopic by considering some examples. Consider a tank with fluid inside and which has fluid entering and leaving through the tank through orifices or pipes as shown below.

For a macroscopic view, we could consider the system to include the boundaries of the tank as shown below. We may define the average density ρ, cross-sectional area A and magnitude v of fluid velocity at each inlet/outlet. We could alternately define a volumetric flow rate (such as gallons per minute) for each inlet or outlet which may be written in terms of fluid density, outlet area and flow velocity.

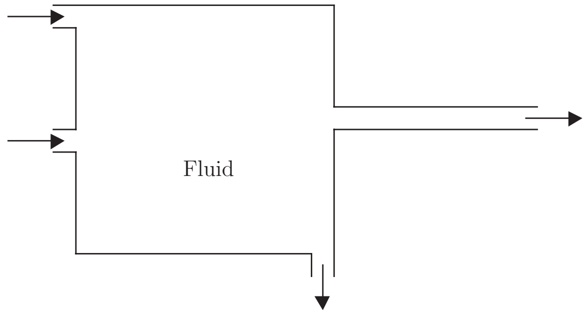


Figure 1.2: Fluid Flow Into and Out of a Tank

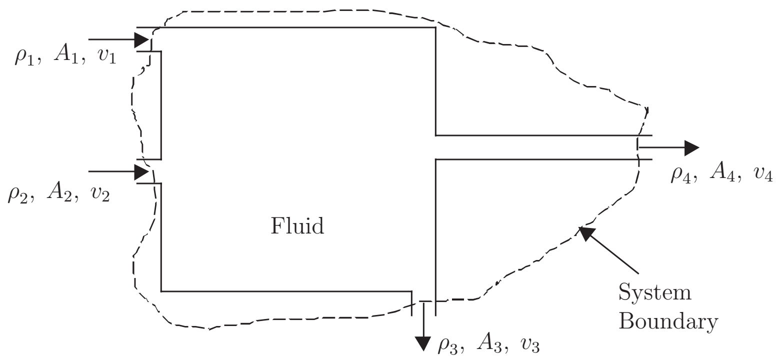


Figure 1.3: Entire Tank Taken as the System

As was done in ENGR 211, we write Conservation of Mass by writing

$$(m_{sys})_{end} - (m_{sys})_{beg} = m_{in} - m_{out} + m_{gen} - m_{con} \tag{1.9}$$

For steady state and no mass generation or consumption, the above reduces to

$$0 = m_{in} - m_{out} \tag{1.10}$$

This can be written in terms of the average fluid density (ρ), magnitude of average fluid velocity (v) and cross-sectional area for each inlet and outlet to obtain:

$$0 = \rho_1 A_1 v_1 + \rho_2 A_2 v_2 - \rho_3 A_3 v_3 - \rho_4 A_4 v_4 \tag{1.11}$$

In the above example, v_1, v_2, v_3, and v_4 are the magnitudes of the average velocity vectors at the corresponding cross-sections. This provides a global or macroscopic picture of conservation of mass for the tank. Notice that it does not take into account any of the *details* of mass flow within the tank (the system) and only takes into account average (or bulk) values of mass flow across the system boundaries (through the inlets and outlets). The macroscopic picture ignores any variation in fluid velocity across the cross-section of the inlets/outlets, it ignores the frictional effects of the fluid on the inlet/outlet orifices, and ignores all the fluid motion within the tank. We would expect that due to the viscosity of the fluid and resultant the frictional forces between the fluid and the tank/orifice walls, the fluid velocity will be smallest near the wall and largest near the center of the orifice.

In order to obtain a more accurate picture of the mass flow within the tank or within the inlet/outlet orifices, we must reduce the size of the system in order to “see” these variations of velocity with position. For example, we could consider any number of smaller systems (subsystem) that include only a small portion of the tank, or one of the inlets or outlets as shown in Figure 1.4:

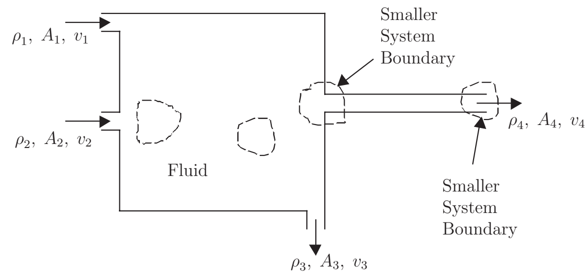
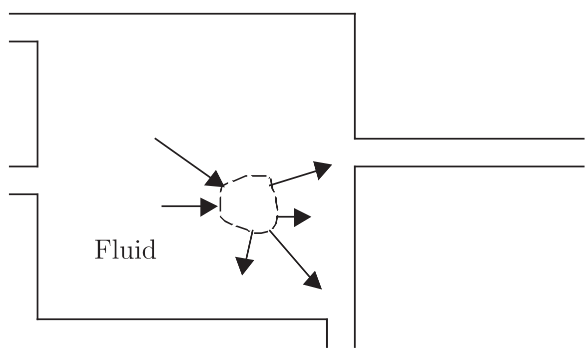


Figure 1.4: Possible “Subsystems” That Could be Chosen

Consider one of the smaller systems in the tank. Fluid will be flowing into some portions of the system boundary and flowing out of other portions of the subsystem boundary as shown below.



Flow in/out of the smaller system

Figure 1.5: Smaller System Chosen Within the Tank

Rather than taking arbitrarily shaped systems as above, we could also define our system to be a regular shape such as a rectangle (in 2-D Cartesian coordinates), a cube (in 3-D Cartesian coordinates), a cylinder (in cylindrical coordinates), or any other “convenient” shape.

By defining the system boundary perpendicular to the coordinate axes, the mathematics will be simplified. We learned this in calculus when the differential area or volume element was chosen carefully to match the coordinate system being used, and the coordinate system was chosen carefully to match the geometry over which integration was to be performed. Furthermore, if we define the system small enough and take the limit mathematically (as in calculus), as its volume goes to zero, we should be able to obtain a Conservation of Mass statement for any point within the body. The resulting local conservation equation will be a differential equation, which, together with appropriate boundary and initial conditions, will completely define the solution. Likewise, conservation of linear and angular momentum will take the form of differential equations with appropriate boundary and initial conditions. The development and solution of these governing differential equations will be covered in some detail in the following chapters.

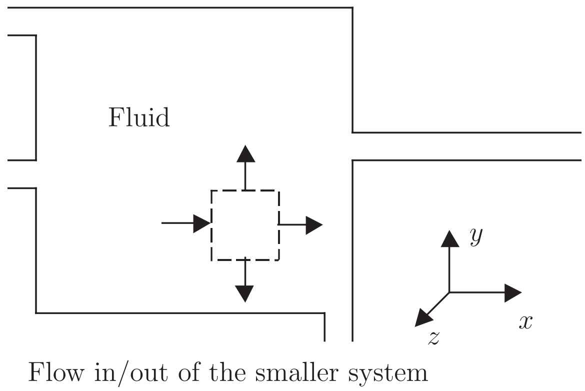


Figure 1.6: Rectangular Subystem (Possibly Differential in Size)

As another example of this detailed flow, consider fluid flow through the exit pipe of the tank example as shown in Figure 1.7 below:

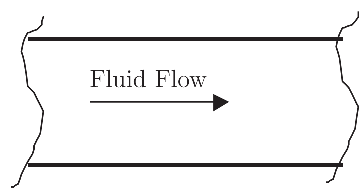


Figure 1.7: Fluid Flow Through a Pipe

If we consider the macroscopic view of the pipe, we can only state the average inlet velocity of the mass flow as it enters or exits the pipe (Figure 1.8a). However, as already discussed it is reasonable to expect (and is confirmed by flow visualization experiments) that the velocity is not constant over the cross-section. Due to the viscous effects, the magnitude of the flow velocity is larger at the center of the pipe as shown in Figure 1.8b.

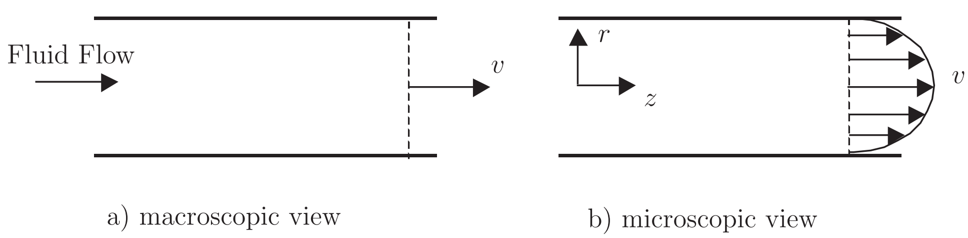


Figure 1.8: Macroscopic and Microscopic Views of Fluid Velocity Profile

Thus the fluid velocity is a function of the radial position r (in a cylindrical coordinate system) and the axial position z (and possibly time if the fluid flow is not steady). Furthermore, in most cases, this velocity function will be a *smooth function of position*, $v = v(r, z)$.

1.2.2 Solids

The motivation for dealing with subsystems in solids is better demonstrated by looking at the conservation of linear momentum. For rigid bodies, the fundamental construction of conservation of linear momentum follows similar reasoning as for conservation of mass except that we must consider the forces applied to the system. In ENGR 211, we idealized rigid truss members as having collinear tensile or compressive forces at their end points (Figure 1.9).

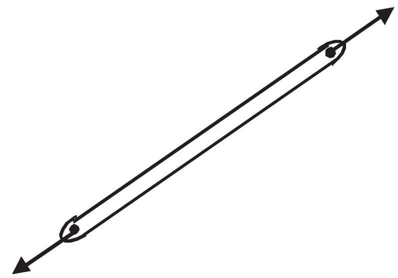


Figure 1.9: Forces on a Two-Force Truss Member

This idealization was reasonable if we were interested only in information about the resultant forces on the member ends and the truss joints or about the reactions at the truss supports.

For frame members, we likewise idealized the forces and moments carried by the member with equivalent axial and shear forces plus bending moments. For example, consider the angled plane frame member shown in Figure 1.10 below:

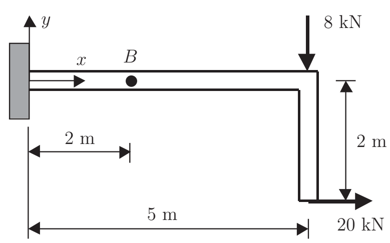


Figure 1.10: Frame Structure With Applied Loads

If we desired information about the forces and moments at point B (located at $x = 2$ m), we defined a system (or free body) such as the one shown in Figure 1.11.

Note that the resultant force components F and V, and the bending moment M, are applied *from* the environment *to* the free body. By applying the conservation of linear momentum ($\sum(F) = 0$) and Angular Momentum ($\sum(M) = 0$) as they were given in ENGR 211, we can determine that the forces and the bending moment at point B are given by:

$$\begin{aligned} F &= 20 \text{ kN (axial force)} \\ V &= -8 \text{ kN (shear force)} \\ M &= 16 \text{ kN m (bending moment)} \end{aligned}$$

Now we consider the notion that all the external forces on this system (free body) are in reality applied over some finite contact area and vary continuously over this area. For example, the total

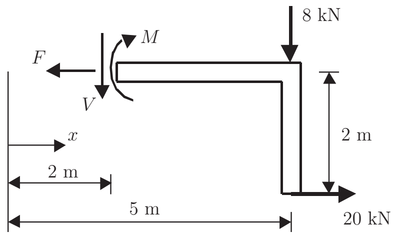


Figure 1.11: Internal Reactions in Frame Member

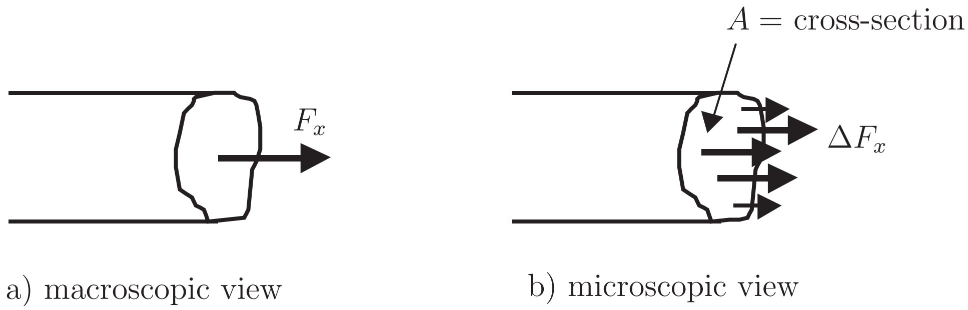


Figure 1.12: Resultant Force (Macroscopic View) and Force Distribution (Microscopic View) at Cross Section of a Frame Member

axial force shown at point B is more accurately represented by a distribution of forces as shown below (note the use of subscript x to indicate the x-component of the axial force):

Suppose that the member has a cross-section A at point x (point B), which is perpendicular to the x-axis. Consider that the area A is divided into small areas and that a force ΔF_x acts on an area ΔA in the x-direction as shown below (ΔA may be arbitrarily shaped or regularly shaped to follow a coordinate system):

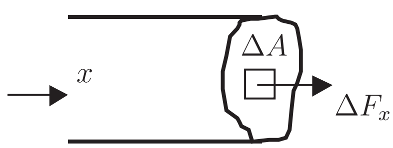


Figure 1.13: Force ΔF_x Acting Over Small Area ΔA

Note that $A = \sum \Delta A$ and $F_x = \sum \Delta F_x$. We define the limit of the quantity $\frac{\Delta F_x}{\Delta A}$ as $\Delta A \to 0$ to be the *normal traction* t_x, where the subscript x denotes a traction component in the x direction. Notice that traction has units of force per area. As we reduce the size of the area (take the calculus limit), we obtain

$$t_x \equiv \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A} \qquad \frac{\text{force}}{\text{area}}. \tag{1.12}$$

If the force varies over the cross-section, then clearly the *normal traction will also be a function over the cross-section.* Consequently, the cross-section will have an equivalent force F_x, but the traction will vary over the cross-section. We may relate the equivalent (resultant) axial force F_x on the cross-section to the traction in the x direction by integrating the traction over the cross-section (again the subscript x is used with the axial force F_x to indicate that this is the x-component of the resultant force):

$$F_x = \int_A t_x dA. \tag{1.13}$$

The shear force in the y direction (V_y) may similarly be related to the traction component in the y direction (*shear traction*) by integrating the traction over the cross-section:

$$V_y = \int_A t_y dA, \tag{1.14}$$

where $t_y \equiv \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A}$, and ΔF_y is the force acting on the area ΔA in the y-direction. In both equations above, the traction components t_x and t_y vary continuously over the cross-section.

Note that since the normal traction t_x varies over the cross-section, this causes a moment about the z-axis located at the *centroid* of the cross-section. Consider the sketch below of a beam where the traction t_x is assumed to vary in the y-direction only.

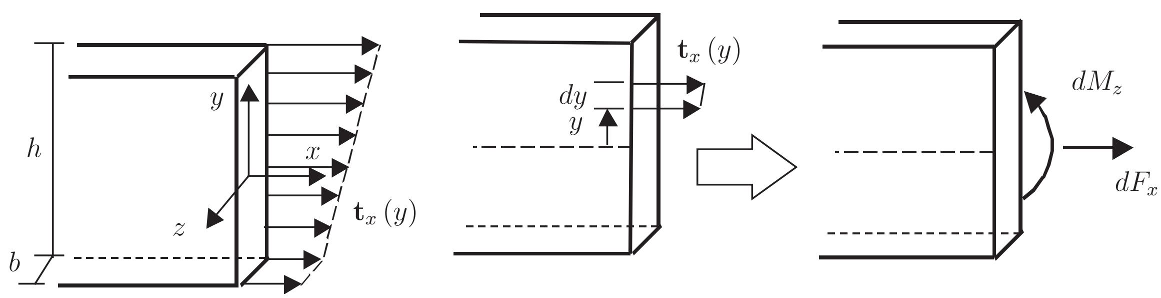


Figure 1.14: Normal Traction Distribution on Beam Cross-Section

The moment about the z-axis due to the traction component t_x acting over a differential area $dA = dydz$ located at some distance y above the x-axis is given by:

$$dM_z = -(t_x dA)y, \tag{1.15}$$

The minus is required due to the sign convention for M_z (right-hand rule). Integrating over the cross-section gives the equivalent moment on the cross-section due to the normal traction distribution:

$$M_z = -\int_A t_x y dA, \tag{1.16}$$

We note again the concept of a *continuum* and the microscopic approach vs. the macroscopic approach. The quantity defined as traction varies with position over the cross section of the beam member, while the resultant forces and moments do not.

For a more complex body, the tractions acting at a point would, in general, have three components. For example, consider a structure such as Rudder Tower shown in Figure 1.15:

Choose an area of the structure for closer examination and define the system to be a differential volume element of size $dx \times dy \times dz$ and located at some position x, y, z. At this point, there would

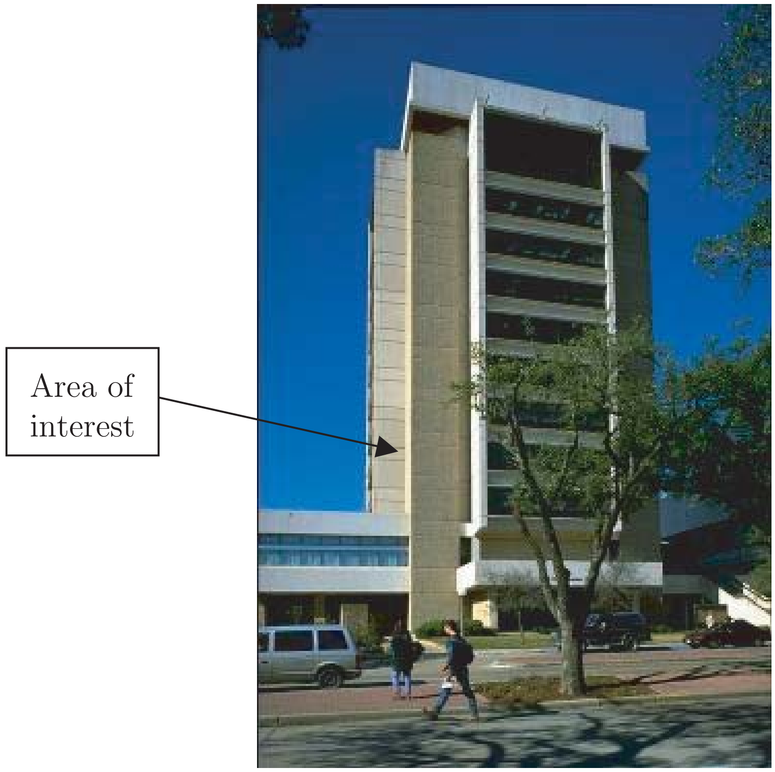


Figure 1.15: Rudder Tower (a Macroscopic View)

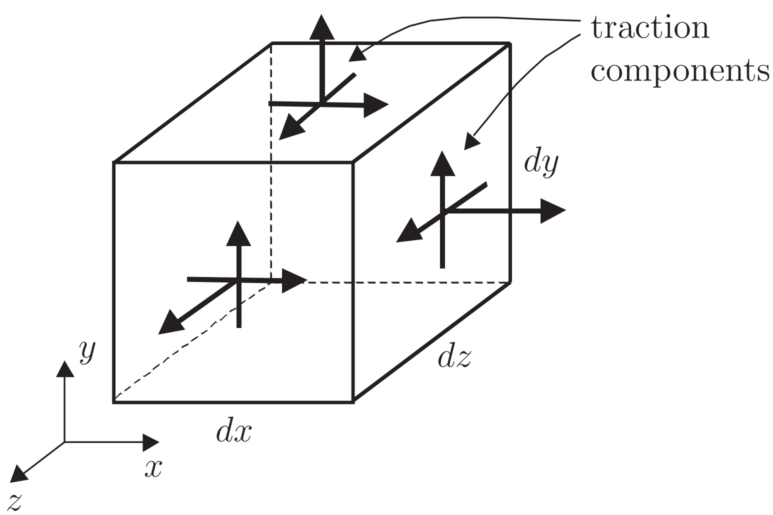


Figure 1.16: Differential Volume Element (System) with Tractions Acting on Boundary

exist tractions on each face of the differential volume (for clarity, only the components of traction on the visible faces are shown):

The concept of traction will be discussed further as the Conservation of Linear Momentum is developed in Chapter 3. It should be noted that tractions are similarly defined for both solid and fluid bodies. An example of traction in a fluid is the hydrostatic pressure that a fluid exerts on a submerged body (which was discussed in ENGR 211).

1.3 This Course

This course will develop the conservation principles for mass, linear momentum, angular momentum, and energy for a continuum. In doing so, the system will be chosen as a differential element and considerable use will be made of calculus and differential equations. Applications will be made to such engineering problems as

- Viscous fluid flow
- Heat conduction in solids
- Deformation and stress distribution for the cases of
  - Bars with axial loads
  - Cylinders with torsional loads
  - Beams in bending

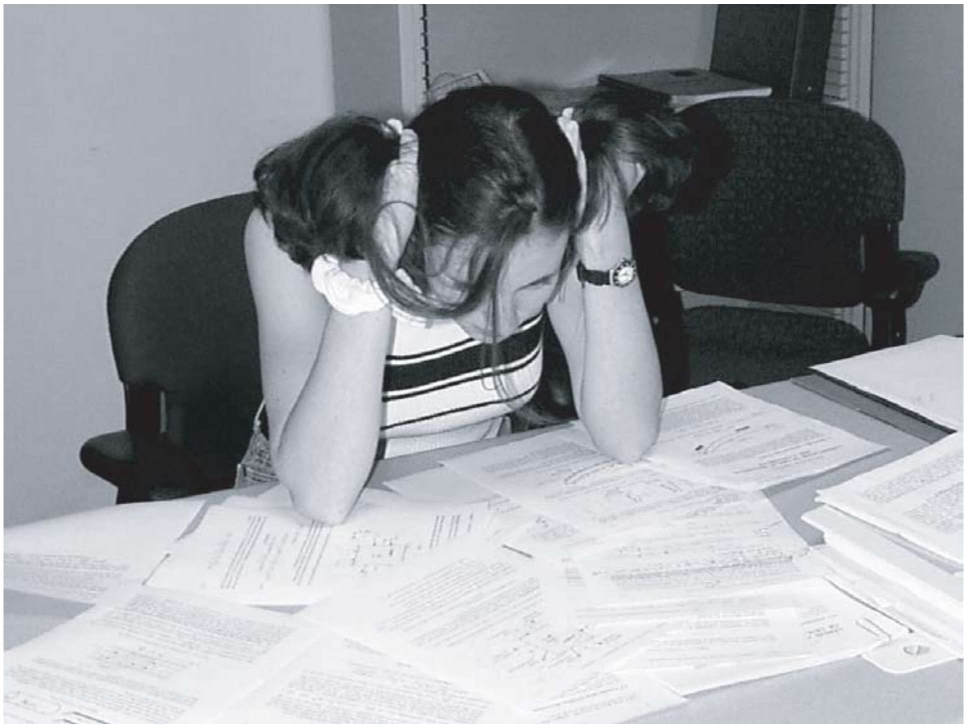
The detailed study of such problems will necessitate that we develop other necessary relations including

- Constitutive relations for fluids and solids. These are relations obtained from experimental observations that relate stress to strain, frictional force to the velocity gradient in a fluid, and temperature to heat flux. For example, in the relation $F = k\delta$ where F is the force on the spring, δ is the extension of the spring, and k is the spring stiffness which is a function of the material the spring is made of and must be experimentally obtained.
- Kinematic relations. These relate deformation to deformation gradients in a deformable continuum body and are obtained from geometric considerations alone. They are similar to kinematic relations used in ENGR 211 for many problems (pulley problems, rigid body dynamics where tangential velocity can be related to angular velocity, etc. wherein the relationship depends solely on the geometry of the particular problem).

1.4 Prerequisite Topics

Included here are typical problems studied in ENGR 211 that form the basis for the understanding of conservation laws for macroscopic systems. As such, they are prerequisite topics that must be understood. The student is urged to work these problems and review any unfamiliar areas before continuing.

Deep Thought

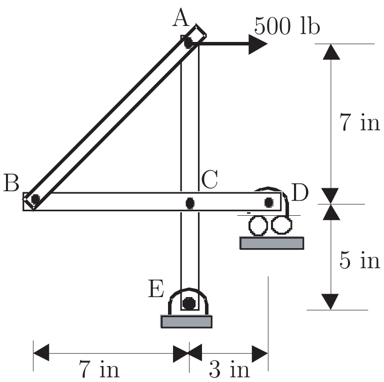


If you wake up one morning with a headache and you think it might have been caused by your troubles in continuum mechanics, consider this solution: dissolve an aspirin in a glass of water and drink it all at once.

Continuity helps relieve pain.

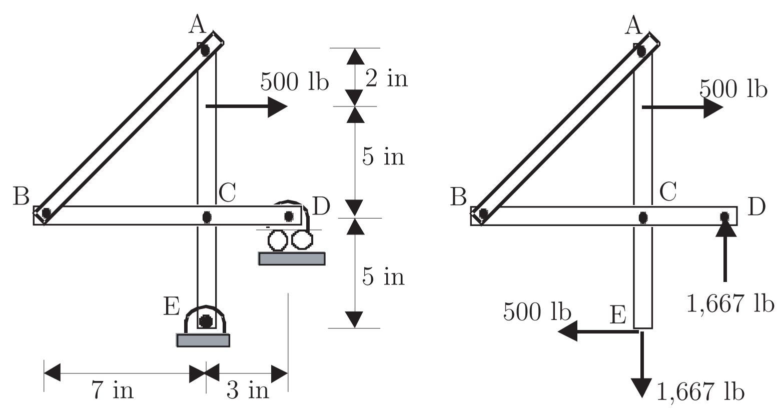
1.5 Problems

1.1 Consider the three-bar frame structure shown at the right. The bars are pinned together at A and B and the structure is supported by a pin support at E and a roller support at D. Determine only the reactions at the supports. Show your results (magnitude and direction) on a separate sketch.



Problem 1.1

1.2 Consider the three-bar frame with the load 2 inches below joint A. The reactions at the supports can be determined to be as shown on the right sketch. Determine the reactions on pins A and C.



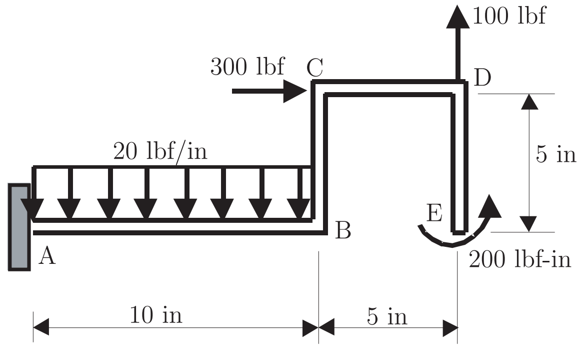
Problem 1.2

1.3 The plane rigid frame is cantilevered at A. Determine

a) Internal reactions (axial, shear and bending moment) at a point 3 inches to the left of point D.

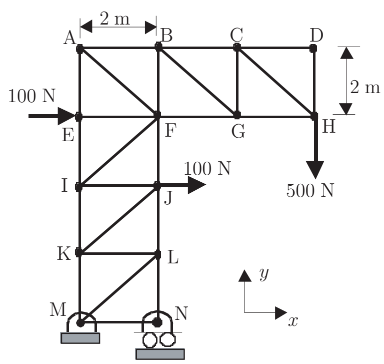
b) Internal reactions at a point 5 inches to the left of point B.

In each case, show your answer (magnitude and direction) on a free body sketch.



Problem 1.3

1.4 Consider the truss structure shown. Typical spans are 2 m×2 m. The reactions are determined to be $M_x = -200$ N, $M_y = -1,500$ N and $N_y = 2000$ N. Determine the force in member IF (and indicate whether it is in tension or compression). To make problem easier to grade put: The reactions are determined to be $M_x = -200$ N, $M_y = -1,500$ N and $N_y = 2000$ N.



Problem 1.4

1.5 An aqueous solution of sodium hydroxide contains 20% NaOH by mass. It is desired to produce an 8% NaOH solution by diluting a feed stream of the 20% solution with a stream of pure water.

a) Calculate the ratios (g pure H_2O / g 20% feed solution) and (g product solution / g feed solution).

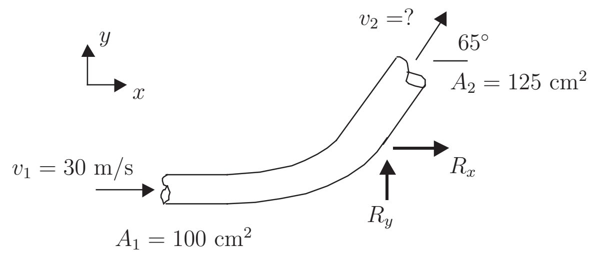
b) Determine the feed rates of 20% solution and pure diluting water needed to produce 2310 $\frac{\text{lb}_\text{m}}{\text{min}}$ of the 8% product solution.

1.6 Fresh water $\left(\text{density} = 1\ \frac{\text{kg}}{\text{liter}}\right)$ flows through a tube which makes a 65° bend in the horizontal plane. At the inlet, the velocity is a steady 30 $\frac{\text{m}}{\text{s}}$ and the cross-sectional area is 100 cm^2; at the outlet the area has enlarged to 125 cm^2. The tube is completely filled with water throughout the bend and the transition in cross-sectional area is smooth.

a) Determine the outlet velocity vector.

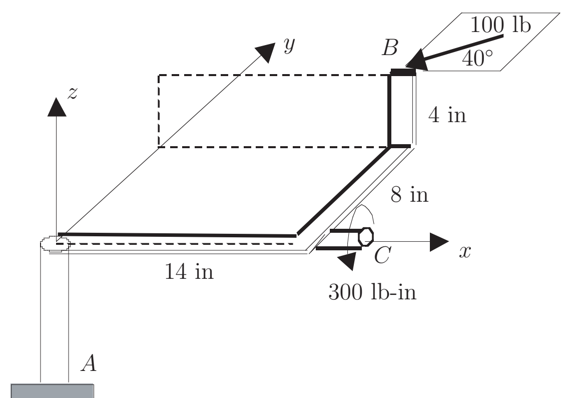
b) Determine the momentum of the fluid crossing the inlet during a time interval of 0.2 seconds.

c) Determine the components of the reactions R_x and R_y, keeping the bend in equilibrium.



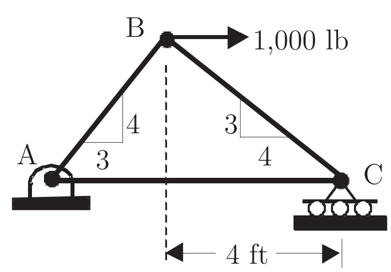
Problem 1.6

1.7 Determine the reactions (forces and moments in cartesian coordinates) at point A. A force of 100 lbs acts at point B (in an x-y plane) and a torque of 300 in-lb acts at C.



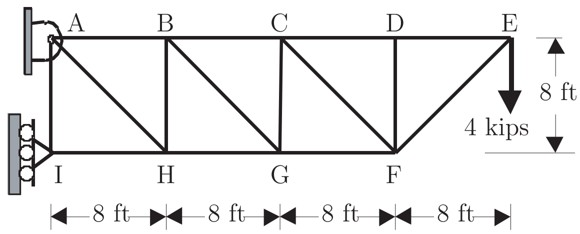
Problem 1.7

1.8 Determine the reactions and force in each truss member (and indicate whether the member is in tension or compression).



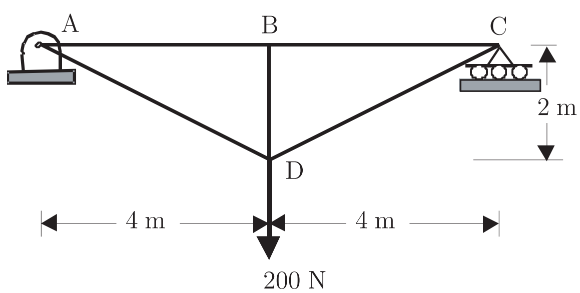
Problem 1.8

1.9 Consider the truss structure which has a pinned support at A and a roller support at I. Determine the force in members EF, BG and HG and indicate whether they are in tension or compression.



Problem 1.9

1.10 The structure to the right is a truss. It has a pinned support at A and a roller support at C. Determine the force in *all* truss members and indicate whether they are in tension or compression.



Problem 1.10

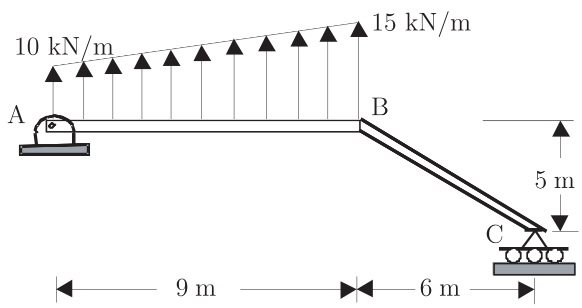
1.11 The frame structure carries the distributed load shown. In addition, the entire structure has a mass of 10 $\frac{\text{kg}}{\text{m}}$ (of member length). Determine the equivalent force(s) on each member (magnitude, direction, and location) due to the applied distributed load and the member weight. DO NOT combine the equivalent distributed load and weight forces. Show your final results on a separate sketch of the structure.

1.12 The frame structure carries the point load and moment shown. There is a pin support at A and a roller support at C. Determine:

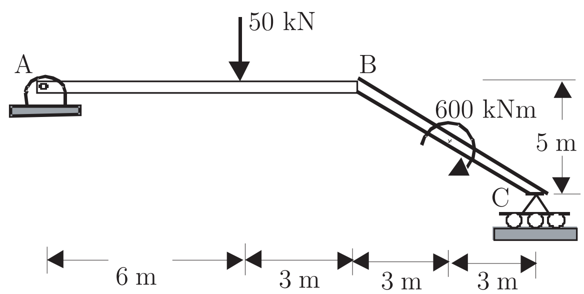
a) Reactions at A and C. Show on a sketch (magnitude and direction).

b) Internal bending moment at B.

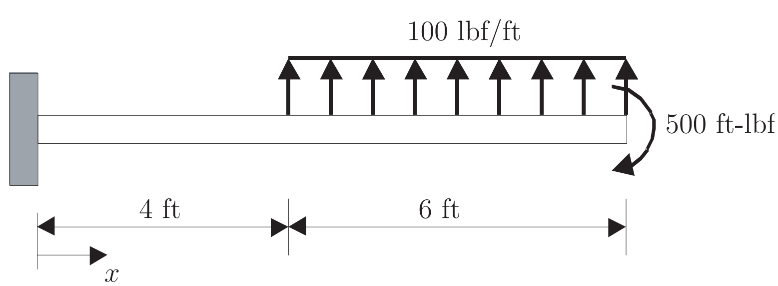
1.13 The cantilever beam is fixed at its left end. Determine the equations for the internal shear $V(x)$ and bending moment $M(x)$ using the coordinate system shown.



Problem 1.11



Problem 1.12



Problem 1.13