Chapter 2

CONSERVATION OF MASS FOR A CONTINUUM

2.1 Continuum Hypothesis

In ENGR 211, we applied conservation laws to systems that are viewed *macroscopically.* In this course, we will learn to apply them to systems that are viewed as *continua*, beginning with this chapter on the conservation of mass.

When we consider a system macroscopically, we summarize the state of the system by assigning physical properties to discrete points in the system and to the system as a whole. (By "assignment" we don't necessarily mean an arbitrary assignment—it might come from a calculation). While variation in position is discrete, variation in time is continuous.

As an example, consider the flow of water through a system consisting of a hose and its two circular openings. We assign to the inlet a mass inflow rate and to the outlet a mass outflow rate. We assume that time progresses continuously. Therefore, if the mass changes smoothly as time progresses (in an informal sense, for now), then these rates are time derivatives.

The rates themselves vary with position (from inlet to outlet), but the variation is not continuous. There is a discrete jump in space between the two positions at which a mass flow rate is defined. Therefore, variations in the flow rate (or any other quantity) from point to point in space cannot be described with space derivatives, because derivatives describe continuous, instantaneous change.

When we consider a system as a continuum, we assign physical properties to every point in the system. In this case, variations in both time and position are continuous.

Consider the water hose again. Instead of assigning an average mass flow rate to the inlet and outlet, we assign a mass flow rate to all points in the planes that define the inlet and the outlet. (Actually, we assign something like "mass density flow"; we'll cover the specifics later.) In fact, we'll assign a flow rate to every point inside the hose. Again, time flows continuously, so these flow rates are time derivatives if the rates change smoothly as time progresses.

The rates themselves again vary with position, but in this case, there is not a discrete jump in space between points at which mass density flow (or any other quantity) is defined. Therefore, if the flow rates (or other quantities) change smoothly as position changes (again, in an informal sense), then we can describe the variations in the flow rates with space derivatives.

Since, in the continuum setting, we will be assigning physical properties to all points in the system, we must examine at every point in the system the conservation laws that apply to those properties. The conservation laws are equations that explain changes in certain physical quantities. In the continuum setting, those changes can be described as derivatives with respect to time and space variables. The conservation laws will thus take the form of partial differential equations in the continuum setting. The introduction of these partial differential equations is at the heart of continuum mechanics.

Now that we have stated roughly what the difference is between the macroscopic view and the continuum view, we will set foundations more formally. A **continuous medium** (or **continuum** or **continuum body**) is defined to be a material system whose total mass is given by

$$M = \int_V \rho(x, y, z, t)\, dV,$$

where the function $\rho(x, y, z, t)$ is the mass density of the medium at the point (x, y, z) and time t. We require that the mass density function be *piecewise smooth*: that the space region in which the system lies can be divided into subregions in which the density function has a continuous derivative with respect to each of the variables x, y, z and t. Further, we require that all quantities of interest be defined as piecewise smooth functions of the space and time variables.

To be able to assign physical properties to the points in a material system in a nonhomogeneous way, the points must be occupied by mass in a "distributed" sense. To this end, we introduce the notion of mass density that is defined at a point in space (x, y, z) and at a time t in such a way that its integral over the material system results in the mass of the system as introduced above. A definition of mass density from which the above relationship can be derived is

$$\rho(x, y, z, t) = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V}, \tag{2.1}$$

sometimes called the **continuum hypothesis**.

The mass density $\rho(x, y, z, t)$ is usually a non-constant function of position and time. For example, consider the mass density of the earth's atmosphere. It varies with position since it decreases with increasing distance from the earth's surface. It also varies from point to point at the same altitude: atmospheric density at sea level may not be the same in Bombay and New York because the weather in these two places is different. Further, atmospheric density varies with time: at neither New York nor Bombay does the weather stay the same throughout the year.

Figure 2.1 shows a schematic representation of the above definition for mass density, with ΔM being the mass contained in a volume ΔV whose centroid at (x, y, z) is pointed to by the vector $\mathbf{r}$. Although ΔV is depicted as a parallelipiped region, the limit causes the shape not to affect the definition of mass density at the **continuum point r**.

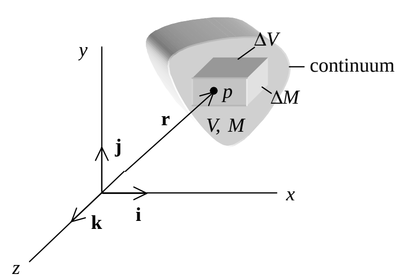


Figure 2.1: Definition of Mass Density at a Point p in a Material System

For the sake of practicality, we must consider the appropriate length scale for defining or measuring a property such as density so that it actually is a piecewise continuous function of position within the body. Is it possible to look too closely at a system so that properties are no longer piecewise continuous? To answer this question, consider an engineering structure such as a bridge, an airplane, an automobile, or a building that includes continuum components that are solid (the structure), fluid (within water pipes) and gaseous (air) that are all functionally connected and contributing to the operation of the structure. The engineering structure shown in Figure 2.2 has as its characteristic length scale the height or width of the building, which is of the order of tens of meters. The continuum components that make up the building have characteristic length scales of the order of centimeters to meters. Within these components we can define the mass density selecting infinitesimal volume elements much smaller in size than this characteristic length, yet larger than atomistic length scales.

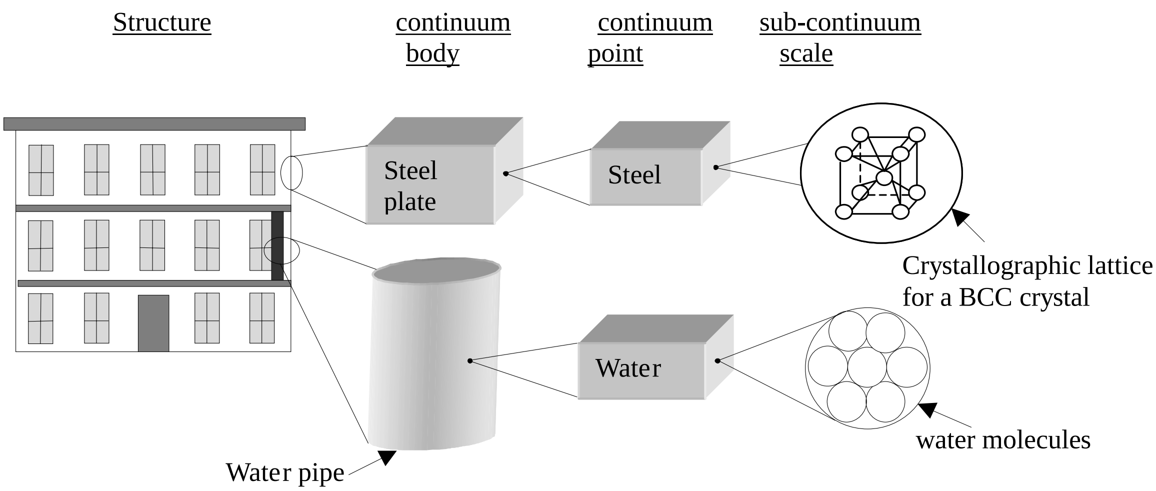


Figure 2.2: Schematic view of an engineering structure, continuum body & continuum point.

To appreciate the importance of the length scale in the definition of mass density and in the continuum concept, consider progressively closer views of various components in the system shown in Figure 2.2. Schematically plot mass density as a function of the volume ΔV that is used in the definition of mass density (it is assumed here that the volume element is proportional to the third power of the characteristic length). For each of the components of an engineering structure for which density can be defined for a wide range of length scales, there will be a length scale for which the discrete structure of the material will result in volume-element-dependent mass density, as Figure 2.3 indicates.

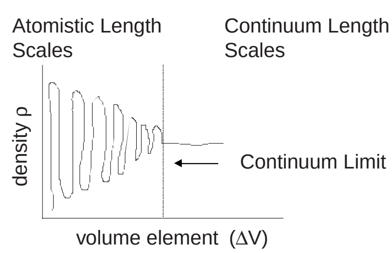


Figure 2.3: Continuum Limit as Defined in Terms of Density

The reason for the variation in ρ as Δ becomes very small is that the mass enclosed by ΔV is not continuously distributed any longer. That is, as we sample the material with ΔV decreasing, we may include a varying number of discrete elementary particles, and the way we choose the size of ΔV has an influence on the value of ρ. Therefore, the continuum limit for a given material system corresponds to the length scale ($\sqrt[3]{\Delta V}$) at which ρ can be defined regardless of the size of ΔV, as

ΔV becomes infinitesimally small. For many engineering materials, we can consider length scales down to the order of 10^{-9} m to be still in the range of a continuum. Below such a critical length scale, we encounter discrete building blocks of the material that cause the density to vary with the size of the volume element. We should mention here that as large-scale engineering structures consist of substructures, which are continuum bodies themselves, many heterogeneous engineering material systems consist of continuum subcomponents, e.g., concrete, composites. Such composite material systems are considered to be continua at different length scales with average or effective mass density defined at engineering length scales (centimeters to meters) and mass densities defined for each of the constituents at smaller length scales (determined by the average size of the aggregate for concrete or fiber diameter in the case of a fibrous composite).

It is emphasized again that after ρ is defined at a continuum point $\mathbf{r}$ and time t by using the above limiting process, then it could be defined the same way at other points in space and time with different numerical values. Therefore, ρ could be a function of $\mathbf{r}$ and t as was mentioned before for the case of the atmosphere. As another example, water is denser at the bottom of an ocean than it is on the surface, and its density may change from season to season. *A continuum body is a collection of continuum points for which the mass density is defined at each point for all time.* As mentioned above, a collection of continuum bodies will form an engineering structure or system. (The word *structure* usually implies application of forces, but in this course, *structure* and *system* are used with the same meaning.)

2.2 Conservation of Mass

Now we will begin the derivation of the conservation of mass equation for continuous media. First, we will derive the conservation of mass for a continuum in which quantities vary in only one spatial dimension. We will then expand the derivation to two and three dimensions in Cartesian coordinates.

2.2.1 Conservation of Mass in 1-D for a Continuum

Consider first the derivation of conservation of mass in one dimension. By this, we mean that the flow field physically and mathematically corresponds to a flow that has only one non-zero velocity component (we arbitrarily choose this to be the x-component), which is independent of the other coordinates. An example of 1-D flow is laminar (smooth, non-turbulent) flow of a non-viscous fluid in a straight pipe, which is assumed to have a "slippery" surface so that the fluid does not "stick" on the pipe wall.

The conservation of mass statement (1.3) requires that we define how much mass passes through the boundary of the system (in this case, the cross-section A_x) during some time period. The easiest way to understand this is to define the **mass flux**, which is the mass flow per unit area per unit time. From a dimensional analysis, we can write

$$\text{mass flux} = \frac{\text{mass flow rate}}{\text{area}} = \frac{\text{mass flow}}{(\text{area})(\text{time})} = \frac{\text{mass}}{\text{volume}} \cdot \frac{\text{length}}{\text{time}} = (\text{density}) \cdot (\text{velocity}) = \rho v_x, \tag{2.2}$$

To see the physical meaning, ask the question "how much mass will flow through a pipe of cross-sectional area A_x in time Δt?" This will be a volume of fluid given by A_x times the distance Δs that the fluid has moved from some reference during a time interval Δt, $\Delta s = v_x \Delta t$, or V = volume of fluid = $A_x v_x \Delta t$. The mass of this fluid volume is given by mass = $\rho V = \rho A_x v_x \Delta t$. The mass flux entering the pipe at the reference point becomes

$$\text{mass flux} = \frac{\text{mass}}{(\text{area})(\text{time})} = \frac{\rho A_x v_x \Delta t}{A_x \Delta t} = \rho v_x, \tag{2.3}$$

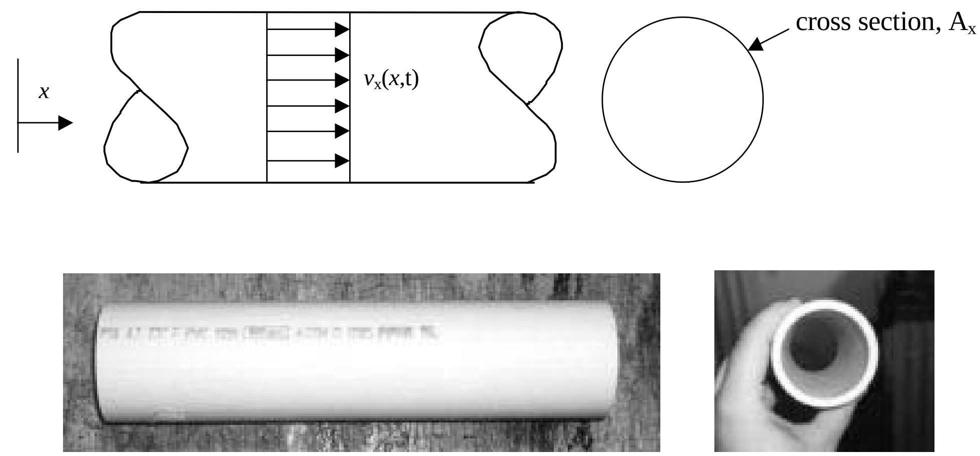


Figure 2.4: Flow of a non-viscous fluid down a straight pipe with a "slippery" wall.

Consequently, we can determine the amount of mass passing through an area (entering or leaving) during a time interval Δt by writing

$$\left(\begin{array}{c}\text{mass entering or}\\ \text{leaving a system}\end{array}\right) = (\text{mass flux})\cdot(\text{area passing through})\cdot(\text{time interval}) \tag{2.4}$$

For the mathematical description of the conservation of mass, consider a plane view of a 1-D flow as shown below. Define A_x to be the cross section of the flow as shown in Figure 2.5, and assume it to be constant. Choose the system to be an infinitesimal control volume of length Δx as shown below, and assume that mass enters and leaves the system during an infinitesimal time interval Δt.

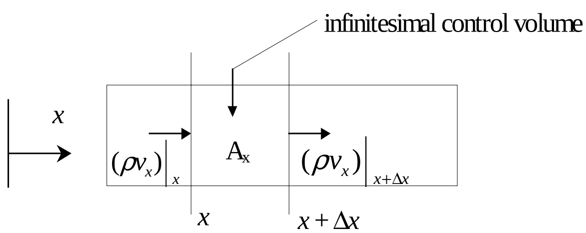


Figure 2.5: Free Body Diagram of infinitesimal control volume for Conservation of Mass in 1-D

To obtain the mass entering the system at point x, use 2.3 to obtain $\left.(\rho v_x)\right|_x A_x \Delta t$. Similarly, the mass leaving the system at point $x+\Delta x$ is given by $\left.(\rho v_x)\right|_{x+\Delta x} A_x \Delta t$. The change in mass within the system during the time period Δt is given by $\left.\tilde{\rho}\right|_{t+\Delta t} A_x \Delta x - \left.\tilde{\rho}\right|_t A_x \Delta x$, where $\tilde{\rho}$ indicates the average mass density in the control volume, which is a function of time $\left(\tilde{\rho}(\,x, \Delta x, t\,) = \frac{1}{\Delta x}\int_x^{x+\Delta x} \rho(\,x, t\,)\,dx\right)$. Substituting these quantities into the conservation of mass statement gives:

$$\left.\tilde{\rho}\right|_{t+\Delta t} A_x \Delta x - \left.\tilde{\rho}\right|_t A_x \Delta x = \left.(\rho v_x)\right|_x A_x \Delta t - \left.(\rho v_x)\right|_{x+\Delta x} A_x \Delta t \tag{2.5}$$

Note that the notation $\left.(\rho v_x)\right|_x$ indicates that the quantity ρv_x is evaluated at the location x. Divide

2.5 by $A_x \Delta x \Delta t$ to obtain

$$\frac{\tilde{\rho}|_{t+\Delta t} - \tilde{\rho}|_t}{\Delta t} = -\frac{(\rho v_x)|_{x+\Delta x} - (\rho v_x)|_x}{\Delta x} \tag{2.6}$$

As the size of the infinitesimal volume element and the length of the time interval go to zero, $\begin{matrix}\Delta t \to 0 \\ \Delta x \to 0\end{matrix}$, we have the one-dimensional conservation of mass equation (also called the **continuity equation**), which is a partial differential equation in x and t:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v_x)}{\partial x} \tag{2.7}$$

Note that both ρ and v_x are functions of x and t ($\rho = \rho(x, t)$ and $v_x = v_x(x, t)$) and that the average mass density $\tilde{\rho}$ in the control volume becomes the mass density at x as the control volume shrinks to zero: $\rho(x, t) = \lim_{\Delta x \to 0} \tilde{\rho}(x, \Delta x, t)$. Note also that the 1-D flow can be visualized to be a flow between two flat plates of infinite extent (channel flow). In this case the cross sectional area A_x could be taken to be the area formed by the distance between the flat plates and of unit depth in the out of the paper direction.

2.2.2 Conservation of Mass in 2-D for a Continuum System in Cartesian Coordinates

Consider a system where the mass flow may be two-dimensional (i.e., a function of two spatial variables). For example, let us revisit the fluid flow shown in Figure 2.4, where now the fluid has non-zero viscosity and the material particles of the fluid "stick" to the wall of the pipe. The actual flow profile will then look like:

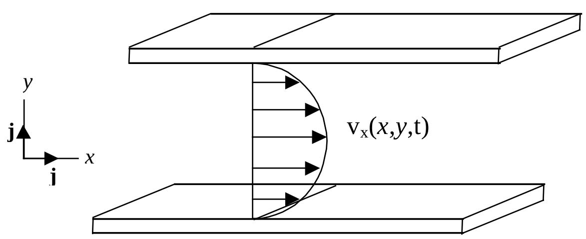


Figure 2.6: Flow of a viscous fluid between two plates with "sticky" walls.

Even though there is still one non-zero component of the velocity, i.e. $v_x \neq 0$, it is a function of two spatial variables (x and y) in addition to time. A 2-D example where both components of velocity (v_x and v_y are non-zero) are the flow in a curved channel which can be visualized as shown in Figure 2.7 (the fluid depth in the z direction is a constant for the channel flow).

If we select a small system within the bend in the shape of a square region, we see that fluid will be flowing into or out of all four boundaries. The velocity vector now has two components: $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$. For any boundary, the only component of flow that is important is that component normal to the boundary. Why? A flow parallel to the boundary does not enter the system while a flow that is at some angle to the boundary will "see" only the normal projection of the area through which to enter.

In order to define conservation of mass for the two-dimensional flow field, we define our system to be an infinitesimal rectangular element of size Δx by Δy. Note that the dimension in the z direction was chosen to be unity since there is no variation in the z direction.

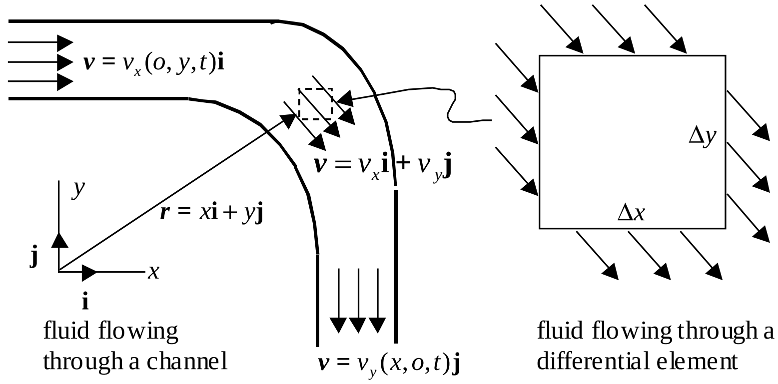


Figure 2.7: Mass Flow in a Channel

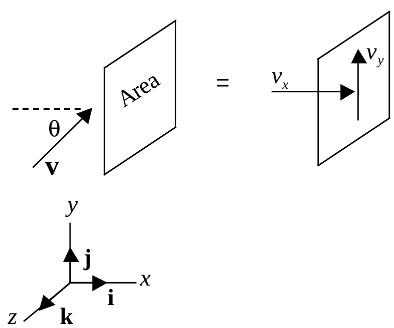


Figure 2.8: Mass Flow Through an Area

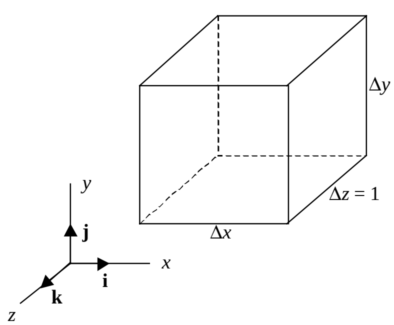


Figure 2.9: Differential Volume Element in Cartesian Coordinates

It is assumed that the mass flux is zero in the z-direction. Consequently, for clarity, we show only the x-y plane view of the fluid flow, together with the above infinitesimal volume element from Figure 2.9.

In order to obtain the mass flow rate through the sides of the rectangular infinitesimal element, we simply multiply the mass flux by the area across which the mass is flowing. For example, assuming Δz

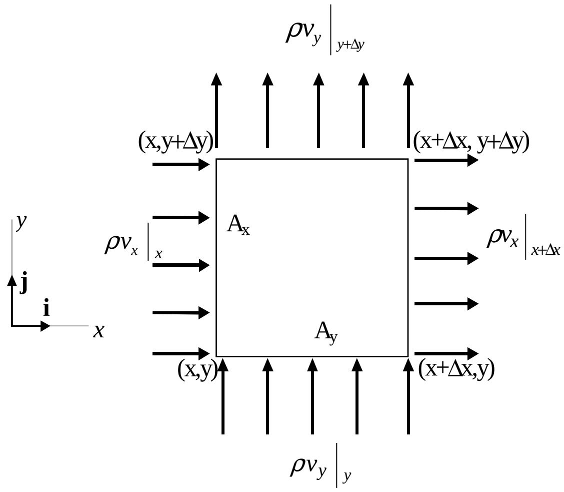


Figure 2.10: Free Body Diagram (infinitesimal control volume) for Conservation of Mass in 2-D

to be an infinitesimal distance in the z-direction (one could choose it to have any value) $\Delta y \Delta z (\rho v_x)|_x$ is the mass flow rate across the surface area $A_x = \Delta y \Delta z$. Similarly, $\Delta x \Delta z (\rho v_y)|_y$ is the mass flow rate across the surface area $A_y = \Delta x \Delta z$. We now use all of these mass flow rates much as we did in the case of 1-D mass conservation, with the following result:

$$\rho|_{t+\Delta t}\, \Delta V - \rho|_t\, \Delta V = (\rho v_x)|_x\, A_x \Delta t - (\rho v_x)|_{x+\Delta x}\, A_x \Delta t + (\rho v_y)|_y\, A_y \Delta t - (\rho v_y)|_{y+\Delta y}\, A_y \Delta t,$$

or

$$\begin{aligned} \rho|_{t+\Delta t}\, \Delta x \Delta y \Delta z - \rho|_t\, \Delta x \Delta y \Delta z &= (\rho v_x)|_x\, \Delta y \Delta z \Delta t - (\rho v_x)|_{x+\Delta x}\, \Delta y \Delta z \Delta t \\ &+ (\rho v_y)|_y\, \Delta x \Delta z \Delta t - (\rho v_y)|_{y+\Delta y}\, \Delta x \Delta z \Delta t \end{aligned} \tag{2.8}$$

where ρ is again the average mass density in the differential control volume element $\Delta V = \Delta x \Delta y \Delta z$ (Δz is taken to be unity).

If we divide both sides of this equation by $\Delta x \Delta y \Delta z \Delta t$ we obtain the following equation:

$$\frac{\rho|_{t+\Delta t} - \rho|_t}{\Delta t} = \frac{(\rho v_x)|_x - (\rho v_x)|_{x+\Delta x}}{\Delta x} + \frac{(\rho v_y)|_y - (\rho v_y)|_{y+\Delta y}}{\Delta y} \tag{2.9}$$

After taking the limits, $\Delta x \Delta y \Delta t \to 0$, we obtain the following partial differential equation in x, y, and t, expressing the Conservation of Mass in 2-D Cartesian coordinates (continuity equation in 2-D):

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y}\right] \tag{2.10}$$

Note: Since the rectangular infinitesimal element is assumed to be at a fixed coordinate location, the representation of conservation of mass with respect to a fixed coordinate system (control volume) is called **Eulerian** representation. The significance of 2.10 is that if it can be satisfied at every point in a material system, then mass must be conserved in that system. Now since it is only one equation, and there are three unknowns: $\rho(x, y, t)$, $v_x(x, y, t)$, $v_y(x, y, t)$, the problem can not be considered adequately defined at this point. Thus, we will need additional conservation equations and, as we will see later, assumptions about the material behavior.

Conservation of Mass in Vector Form

Equation 2.10 can also be written in vector form as

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \tag{2.11}$$

where ∇ is the divergence operator. Using the representation of ∇ in Cartesian coordinates, we have the following explicit evaluation for the right side of the conservation of mass equation 2.11:

$$\begin{aligned} \nabla \cdot (\rho \mathbf{v}) &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} \right) \cdot (\rho v_x \mathbf{i} + \rho v_y \mathbf{j}) \\ &= \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) \end{aligned} \tag{2.12}$$

which is exactly the negative of the right side of 2.10.

The condensed vector notation can be used to write the conservation of mass equation for *any coordinate system* and is valid for 1-D, 2-D or 3-D cases:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \tag{2.13}$$

1-D Case: $\mathbf{v} = v_x \mathbf{i}$

$$\begin{aligned} \nabla \cdot (\rho \mathbf{v}) &= \left(\mathbf{i}\frac{\partial}{\partial x} \right) \cdot (\rho v_x \mathbf{i}) \\ &= \frac{\partial}{\partial x}(\rho v_x) \end{aligned}$$

2-D Case: $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j}$

$$\begin{aligned} \nabla \cdot (\rho \mathbf{v}) &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} \right) \cdot (\rho v_x \mathbf{i} + \rho v_y \mathbf{j}) \\ &= \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) \end{aligned}$$

3-D Case: $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$

$$\begin{aligned} \nabla \cdot (\rho \mathbf{v}) &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial x} \right) \cdot (\rho v_x \mathbf{i} + \rho v_y \mathbf{j} + \rho v_z \mathbf{k}) \\ &= \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) \end{aligned}$$

2.2.3 Conservation of Mass in 3-D for a Continuum in Cartesian Coordinates

The system is now defined to be an infinitesimal parallelepiped element, as shown in Figure 2.11:

Following an approach similar to that which we used in the 2-D case, we obtain the following partial differential equation representing Conservation of Mass (continuity equation) in 3-D Cartesian coordinates:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z} \right] \tag{2.14}$$

where all four quantities $\rho(x, y, z, t)$, $v_x(x, y, z, t)$, $v_y(x, y, z, t)$ and $v_z(x, y, z, t)$ are functions of position and time.

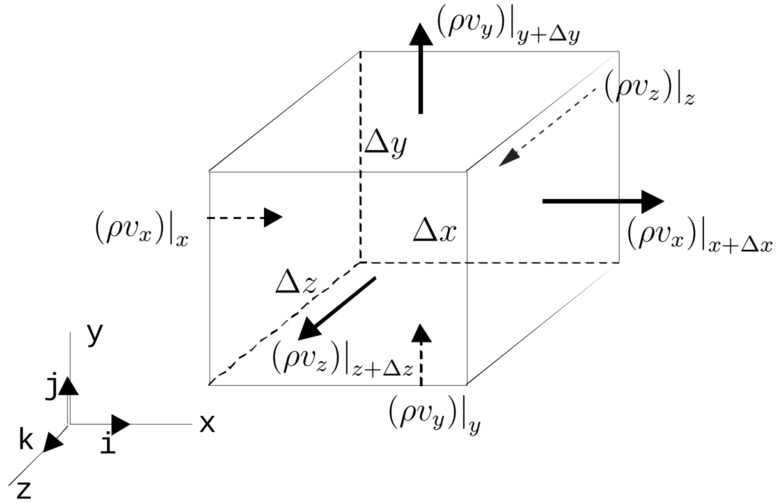


Figure 2.11: Free Body Diagram (infinitesimal control volume) for Conservation of Mass in 3-D

In vector form, 2.14 can be written as

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \tag{2.15}$$

In 2.15 above, $\frac{\partial \rho}{\partial t}$ is the accumulation rate per unit volume of mass at a point fixed in the coordinate system, and is the net mass flow rate per unit volume into a differential volume at a fixed time. Both ρ and $\mathbf{v}$ can be functions of x and t.

It is important to state once again that 2.15 is applicable to any coordinate system (Cartesian, cylindrical, spherical, etc.). To obtain conservation of mass in a particular coordinate system, we need only use the del operator for that coordinate system. Consequently, the vector representation 2.15 is a very general and powerful representation for conservation of mass. In many disciplines, equation 2.15 is called the **continuity equation**. Equation 2.15 is a statement of the relationship between density and velocity that must be satisfied for all continuous media (hence the name continuity equation).

Several special cases of the conservation of mass are noteworthy. In order to discuss those, let us first consider two new concepts. First, we say that a continuum is at **steady state** if none of the state variables depend on time. Thus, for steady state conditions the conservation of mass reduces to

$$0 = \nabla \cdot (\rho \mathbf{v}) \tag{2.16}$$

A continuum body is **incompressible** if its density in the neighborhood of each material particle cannot be changed under applied loading as it moves. Equation 2.15 may also be written (using vector notation and rearranging terms) as

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = -\rho(\nabla \cdot \mathbf{v}) \tag{2.17}$$

But since the left hand side of equation (2.17) represents how the density in the neighborhood of a material particle changes as it moves (material derivative), 2.17 reduces for an incompressible fluid to

$$0 = -\rho(\nabla \cdot \mathbf{v}) \tag{2.18}$$

Thus, in the incompressible case 2.18 simplifies to

$$0 = \nabla \cdot \mathbf{v} \tag{2.19}$$

Finally, if a continuum is both incompressible and at steady state, it follows that

$$0 = \mathbf{v} \cdot \nabla \rho \tag{2.20}$$

2.2.4 Conservation of Mass in Cylindrical Coordinates

In cylindrical coordinates, the conservation of mass has the following form:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{1}{r}\frac{\partial(\rho v_r r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z}\right] \tag{2.21}$$

Example 2.1

Laminar flow between two parallel flat plates (Poiseuille flow)

Poiseuille flow is the case of fluid flow between two fixed parallel plates separated by a distance d with a pressure gradient in the x direction. The driving force is the pressure differential from left to right. Assume the flow is steady and incompressible. Determine the fluid velocities. Assume the

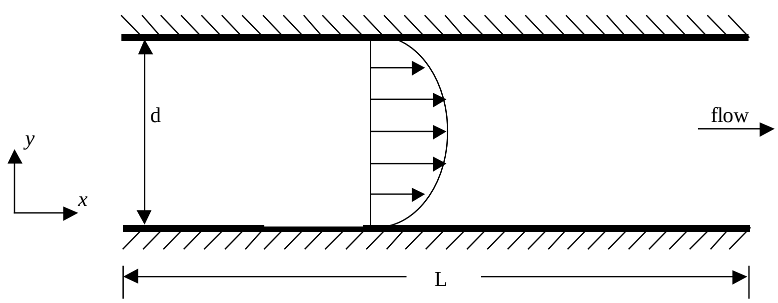


Figure 2.12:

flow is steady state and incompressible:

$$\begin{aligned} \text{steady state} \quad &\Longrightarrow \quad \frac{\partial}{\partial t} = 0 \\ \text{incompressible} \quad &\Longrightarrow \quad \rho = \text{constant, or } 0 = \nabla \cdot \mathbf{v} \text{ (see equation 2.19)} \end{aligned}$$

Assuming that there is no fluid flow normal to the plate (y direction) or in the z direction, the kinematic boundary conditions are written

$$v_y = v_z = 0$$

Conservation of mass (continuity equation) is given by

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right]$$

which reduces to

$$\frac{\partial(\rho v_x)}{\partial x} = 0 \quad \text{or} \quad \frac{\partial v_x}{\partial x} = 0 \implies v_x = f(\,y, z\,) + C_1$$

The statement $\frac{\partial v_x}{\partial x} = 0$ indicates that v_x is not a function of x and is therefore only a function of y and z. For plane motion, the fluid velocity v_x is assumed to be a constant in the z direction and therefore independent of the z coordinate, i.e., $v_x \neq v_x(z)$. Thus, the final solution for v_x from conservation of mass is given by

$$v_x = v_x(y)$$

At this point, there is nothing further that can be established about the actual variation of the velocity distribution v_x with position y (i.e., between the top and bottom plate). In order to complete the solution, additional equations are required. These include conservation of linear momentum (which will be considered in Chapter 3) and a "constitutive equation" which provides information about the fluid material itself (in particular, its viscosity). With these additional equations, we will be able to develop an expression for v_x in terms of the pressure gradient from left to right and the fluid viscosity. We will also be able to show that the fluid velocity v_x has a parabolic profile between the two plates for a viscous fluid.

Example 2.2

Laminar flow through a cylindrical tube

Consider the case of smooth laminar fluid flow through a cylindrical tube. Determine the velocity distribution of fluid in the tube. We begin with conservation of mass in cylindrical coordinates:

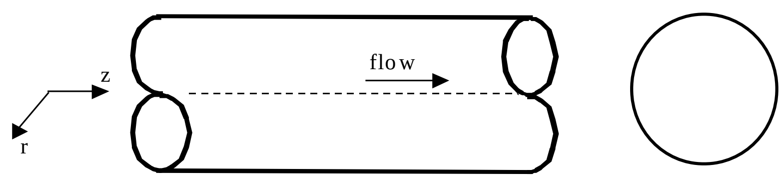


Figure 2.13:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z}\right]$$

Assumptions: We assume that the fluid flow is 1-D in the axial direction of the pipe (z direction) so that $v_r = v_\theta = 0$. Therefore, conservation of mass reduces to

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v_z)}{\partial z}$$

For constant density (incompressible fluid) $\frac{\partial v_z}{\partial z} = 0 \implies v_z = v_z(r, \theta)$

This indicates that v_z does not depend on z and is therefore at most a function of r and θ. It is reasonable to assume angular symmetry (v_z does not depend on θ so that the solution for v_z is a function of r only:

$$v_z = v_z(r)$$

As in the previous example, we can carry the solution no further without additional equations (conservation of momentum) and information about the fluid (constitutive equations).

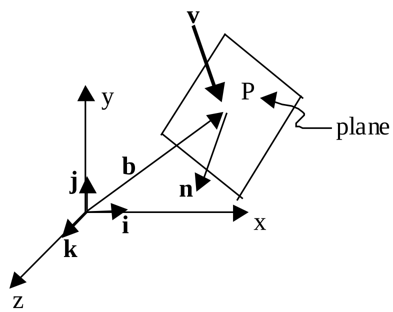


Figure 2.14:

Example 2.3

A fluid flows with velocity $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}\ \frac{\text{m}}{\text{s}}$ at point P having coordinates $(1, 2, 4)$. Consider a plane through P which is normal to $\mathbf{n} = -\mathbf{i} + 2\mathbf{k}$.

a) What is the velocity at which the fluid crosses the plane?

b) If the fluid is water, what is the mass flow rate across 0.1 m^2 of this plane, assuming $\mathbf{v}$ does not vary across the 0.1 m^2?

Solution
The velocity vector and the normal to the plane are

$$\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}; \qquad \mathbf{n} = -\mathbf{i} + 2\mathbf{k}$$

The unit normal vector to the plane is given by:

$$\mathbf{n} = \frac{\mathbf{n}}{|\mathbf{n}|} \qquad \mathbf{n} = \frac{-\mathbf{i} + 2\mathbf{k}}{\sqrt{5}}$$

The velocity at which the fluid crosses the plane is the normal component of the velocity vector, which is obtained by forming the dot product of the velocity vector with the unit normal vector to the plane:

$$v_n = \mathbf{v} \cdot \mathbf{n} = \frac{-2}{\sqrt{5}} \frac{\text{m}}{\text{s}} = -0.894 \frac{\text{m}}{\text{s}}$$

The mass flow rate through an area A is given by:

$$\dot{m} = \text{mass flow rate through area } A = \rho v_n A$$

For water, $\rho = 1000\ \frac{\text{kg}}{\text{m}^3}$, and $\dot{m} = \left(1000\ \frac{\text{kg}}{\text{m}^3}\right)\left(-0.894\ \frac{\text{m}}{\text{s}}\right)\left(0.1\ \text{m}^2\right)$ or, $\dot{m} = -894\ \frac{\text{kg}}{\text{s}}$. Note that the minus sign in the normal component of the velocity indicates that the mass flow is in the direction opposite to the direction of the normal vector to the plane.

Deep Thought

When I was young, I used to dig holes in the sand at the beach thinking I was removing something. Later the tide would fill in my hole. I learned that conservation of mass will come back to haunt me .

2.3 Questions

2.1 In expressing the principle of conservation of mass, what is the difference between the statement for continuous media and that used for macroscopic processes (ENGR 211 versus ENGR 214)? Do the two statements differ? If yes, how?

2.2 Describe in your own words the concept of a continuum or a continuous medium.

2.3 What is the difference between a continuum problem and a system (non-continuum) problem?

2.4 Write the equation of conservation of mass in Cartesian coordinates, and explain the physical meaning of each term.

2.5 In the context of the principle of conservation of mass, what do we mean by incompressible and steady state flows?

2.6 How does the mathematical statement of conservation of total mass change given steady-state conditions?

2.7 Explain the difference between *mass flux* and *mass flow rate*. What would be the appropriate SI units for these quantities?

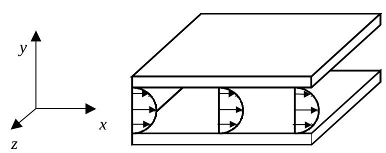
2.8 Explain the basic procedure used to derive Conservation of Mass in 3-D.

2.9 Give the continuity equation for:

  a) A steady-state medium.
  b) An incompressible medium.

2.4 Problems

2.13 Describe in your own words what continuity says about the flow between two parallel plates (steady state, incompressible, 1-D)? Give a real world example.



Problem 2.13

2.14 Define what the mass density is for a building.

2.15 A liquid substance is flowing at a rate of 1200 $\frac{\text{lb}_\text{m}}{\text{s}}$ through a tube with cross-sectional area 50 ft^2. The mass density of the liquid is 60 $\frac{\text{lb}_\text{m}}{\text{ft}^3}$. Apply the Conservation of Mass in 1-D to determine the velocity of the fluid.

2.16 The same liquid in Problem 2.15 is now flowing with a velocity in the x-direction of 2 $\frac{\text{m}}{\text{s}}$. Apply the Conservation of Mass in 1-D to determine the mass flow rate of the substance.

2.17 Water flows with a velocity given by $\mathbf{v} = (5x^2 + y)\mathbf{i} + (3z - 4y)\mathbf{j} + (6az + 5y)\mathbf{k}\ \frac{\text{ft}}{\text{s}}$. Assume that the water flows in steady state and that the mass density is constant. Find a so that conservation of mass is satisfied.

2.18 Water flows with velocity $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}\ \frac{\text{m}}{\text{s}}$ at a point P having coordinates $(2, 1, 3)$. Suppose a plane with normal $\mathbf{b} = 3\mathbf{i} - 2\mathbf{k}$ passes through P.

a) The fluid flows over the plane with what speed?

b) What is the angle between the direction of the fluid and the normal to the plane at P?

c) What is the mass flow rate across 0.1 m^2 of this plane, assuming $\mathbf{v}$ does not vary across the plane?

2.19 A fluid flowing with velocity $\mathbf{v} = 3\mathbf{i} - 7\mathbf{j}\ \frac{\text{ft}}{\text{s}}$ at point Q has coordinates $(2, 3, 5)$. Consider a plane through Q whose normal vector is $\mathbf{n} = -2\mathbf{i} + 3\mathbf{k}$.

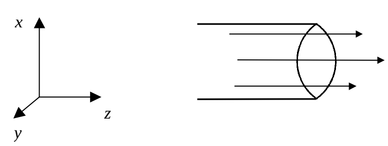
a) The fluid flows over the plane with what speed?

b) If $\mathbf{v}$ is constant and the fluid is water, what is the mass flow rate across 0.1 ft^2 of this plane?

c) Find the mass flow rate at point Q $(2, 3, 5)$ across 0.1 ft^2 of the surface defined by $f(x, y, z) = z^2 + x + y^2 + 16$?

d) Sketch the surface, f, $\mathbf{n}$ (its unit outward normal) and $\mathbf{v}$.

2.20 A fluid is flowing in one direction, axially, in a cylindrical tube of uniform cross-sectional area as shown below:



Problem 2.20

1) The mass density, ρ, is constant

2) The flow is under steady-state conditions

3) The components of the velocity of the fluid, $\mathbf{v}$ are given by:

$$v_x = 0 \qquad v_y = 0 \qquad v_z = v_{\max}\left[1 - \left(\frac{x^2 + y^2}{R^2}\right)\right] = v_{\max}\left(1 - \frac{r^2}{R^2}\right)$$

4) $v_{\max}$ is the maximum velocity of the fluid

5) R is the inner tube diameter

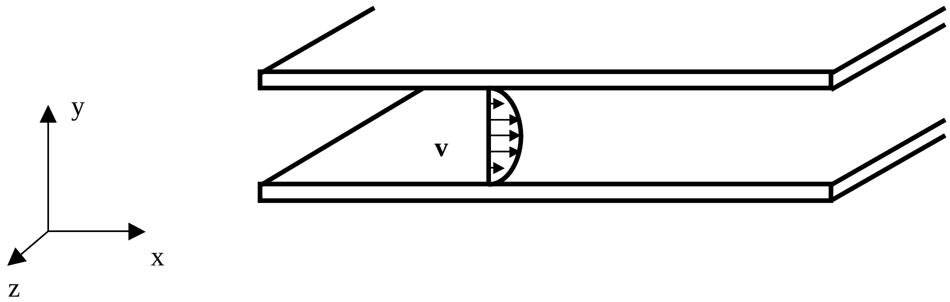
Verify that Conservation of Mass both in Cartesian and cylindrical coordinates is satisfied at every point in the tube.

2.21 A flow between two parallel plates is given as shown. The components of the velocity of the fluid are

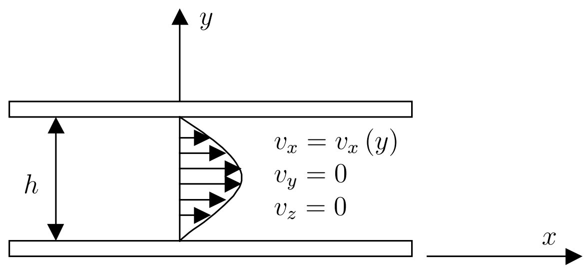
$$v_x = 3y^3 + 2y^2 - 5\ \frac{\text{m}}{\text{s}} \qquad v_y = v_z = 0$$

1) Apply Conservation of Mass to this problem.

2) Is this a steady state problem? Why or why not?



Problem 2.21



Problem 2.22

2.22 The velocity profile for plane flow between two plates is given below. Assume that both plates are stationary. The fluid has depth h (see figure). Let v_x be given by

$$v_x = \frac{1}{2\mu}\left(-\frac{dP}{dx}\right)h^2\left[\frac{y}{h} - \left(\frac{y^2}{h^2}\right)\right]$$

where μ and $\frac{dP}{dx}$ are both constant. Verify that Conservation of Mass is satisfied.